Option Pricing and Hedging with Regret Optimisation

Zhikang Xu

PhD

University of York
Mathematics

March 2018

Abstract

This thesis focuses on the option pricing and hedging based on a regret optimisation problem in a discrete-time financial market model with proportional transaction costs. In such model, the no-arbitrage price interval can be very large. Such large interval makes it difficult for an investor to choose the “right” prices, which is a long standing difficulty in the field. We introduce an indifference pricing method based on minimising regret/disutility, and show that the spread between the buyer’s and seller’s prices can be much narrower than the no-arbitrage price interval. The regret optimisation problem allows possible fund injection/withdrawal at each time step, and in doing so it extends the classic utility maximisation problems in financial models. Moreover, by allowing the investor’s preference towards risk to be different at different time step, it also extends the optimal investment and consumption problem in financial market models with a finite horizon. In addition, the investor’s endowment that is considered in our setting is modelled by a portfolio flow which extends the notion of initial wealth. We prove that there exists a solution to the regret optimisation problem, and indifference prices are always within the no-arbitrage price interval. Under an exponential type regret function, we find a dynamic programming algorithm to construct a solution to a Lagrangian dual problem. By solving the dual problem, we can not only solve the regret optimisation problem but also calculate the option indifference prices. In binary models, we calculate the optimal injection/withdrawal strategy for various different values of given parameters, and also compute the indifference prices of various European options. The numerical results show that the bid-ask indifference price interval can be much narrower than the no-arbitrage price interval, and such smaller price interval can be used to guide the investor to choose the “right” prices.

Contents

List of Tables

List of Figures

Acknowledgements

I am deeply grateful to my supervisors, Dr. Alet Roux and Professor Tomasz Zastawniak, for their invaluable assistance, immeasurable support and inspiration throughout the past four years. My thanks also go to the Department of Mathematics at the University of York for their financial support in the form of a graduate teaching studentship.

Last but not least, I want to thank my family and my wife Manli for their endless love and everlasting support.

Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1

Introduction

Pricing and hedging of derivative securities in financial market models are two of the main topics in modern mathematical finance research. In friction-free complete market models, European options can be replicated by self-financing trading strategies. The fair price of a European option is the initial value of the replication strategy. This replicating and pricing method was pioneered by Black & Scholes (1973) and Merton (1973) who considered pricing and hedging of European call options in a continuous time market model. They provide a closed form formula for the fair prices of European call options in this model. For an overview of pricing and hedging in friction-free market models, see textbooks Bingham & Kiesel (2004) and Föllmer & Schied (2011, Chapter 1.5) (especially in discrete time models) and the literature within it.

In market models with transaction costs, the first main hedging method that is widely used is superhedging. This hedging method is generally more preferable than the replicating method, because it is generally less expensive; for example Bensaid, Lesne, Pagès & Scheinkman (1992), Cutland & Roux (2012, Example 8.29). Similar to replicating, superhedging is independent of an investor's preference. For the option seller, the objective of superhedging is to construct a strategy, with initial value as small as possible, that will enable him to meet his obligation in the option contract at expiry time. Similarly, the option buyer's superhedging objective is to find a strategy which generates the highest amount of bond/cash at the initial time, and at the same time allows him to remain solvent after receiving the payoff of the option. Superhedging provides a method which allows the seller and buyer to hedge without any risk. Works in superhedging include Bensaid, Lesne, Pagès & Scheinkman (1992), Edirisinghe, Naik & Uppal (1993), Jouini & Kallal (1995), Perrakis & Lefoll (1997), Kabanov & Stricker (2001), Delbaen, Kabanov & Valkeila (2002), Dempster, Evstigneev & Taksar (2006), Roux (2006), Roux, Tokarz &

Zastawniak (2008), Löhne & Rudloff (2014), Roux & Zastawniak (2016).

Drawbacks of superhedging include that the bid-ask interval can be very wide; see numerical examples in Roux (2006, Chapter 3.6). This is not very helpful in guiding an investor on choosing "right" prices. In addition, this method does not allow an investor to take any risk. For these reasons, researchers have studied other hedging methods which allow an investor to take risks by taking into account his preference. The objectives of such methods include the maximisation of expected utility of terminal wealth, maximisation of expected utility of consumption during trading, minimisation of expected shortfall risk and other risk minimisation. Relating to existing literature, regarding maximisation of expected utility from terminal wealth, the relevant studies include Hodges & Neuberger (1989), Dumas & Luciano (1991), Cvitanić & Karatzas (1992, 1996), Davis, Panas & Zariphopoulou (1993), Gennotte & Jung (1994), Clewlow & Hodges (1997), Monoyios (2003, 2004), Sass (2005), Zakamouline (2005, 2006), Atkinson & Quek (2012), Kallsen & Muhle-Karbe (2015). Related works in maximisation of expected utility of consumption during trading are Davis & Norman (1990), Cvitanić & Karatzas (1992), Constantinides & Zariphopoulou (1999), Øksendal & Sulem (2002), Liu (2004), Muthuraman & Kumar (2006), Muthuraman (2007), Kallsen & Muhle-Karbe (2010), Hobson & Zhu (2016). Minimisation of expected shortfall risk is studied by Guasoni (2002), and minimisation of local risk is discussed by Mercurio & Vorst (1997), Lamberton, Pham & Schweizer (1998). In the preference based hedging approaches mentioned above, an investor with a given financial endowment controls a trading strategy in order to achieve a hedging objective which depends on his preference. Thus, these methods do consider investors' preferences towards risks. Regarding the pricing of options, a number of utility based optimisation problems are closely related to indifference pricing. In general, the indifference price of an option contract is defined as the price such that the investor would have the same expected utility by entering this contract as by not doing so. Compared to superhedging, the indifference pricing could possibly produce a smaller bid-ask spreads of European options; see Pennanen (2014, Theorem 6) for example. The pricing method considered in this thesis is similar to the indifference pricing based on utility maximisation problems.

The option prices considered in this thesis relies on an optimisation problem. The background of this optimisation problem is related to (1) convex stochastic dynamic programming; and (2) convex-valued random dynamical systems. The study of these two areas became prominent in the context of mathematical economics in the 1960s and the 1970s. Some of the important ideas in these two areas can be traced back as early as von Neumann's work in the 1930s on economic dynamics (see von Neumann (1937)), as well as Kantorovich and Koopmans's studies during the late 1930s to the 1950s on optimal resource allocation and linear programming (see Kantorovich (1960) and Koopmans (1951)).

Regarding convex stochastic dynamic programming, its economic applications are mainly in the area of utility maximisation over a family of admissible investment strategies. For example, Dynkin (1972) (see also Dynkin & Yushkevich (1979, Chapter 9)) made a seminal contribution in this field. Then Arkin & Evstigneev (1987) presented a systematic and comprehensive study of the corresponding theory from deterministic case to stochastic case. The optimisation problem studied in this thesis follows the tradition of utility maximisation. The utility maximisation problem in economics appears in a very general setting and involves modelling of various economic activities, whereas this thesis focuses on a specific financial model. However, these problems share a common objective, namely, achieving the investor's goal by controlling his trading strategies.

Regarding convex-valued random dynamical systems, von Neumann (1937) and Gale (1956) produced pioneering work on its applications in models of economic growth. Then option pricing and hedging under proportional transaction costs have been studied by Dempster, Evstigneev & Taksar (2006), who developed a general framework including trading constraints, and Evstigneev & Zhitlukhin (2013), who studied risk-acceptable hedging in interconnected financial models.

This thesis focuses on the option indifference pricing based on a regret optimisation problem in a two-asset market model with proportional transaction costs. In this regret optimisation problem, an investor faces the liability of delivering a sequence of portfolios. Additionally, at each time step, he needs to manage his financial position in the underlying assets (cash and stock). In this problem, the investor's trading strategy is not required to be self-financing, in other words, he is allowed to inject extra cash beyond the given initial endowment. In each trade, the investor will use a regret function (which needs to be nondecreasing and convex) to evaluate his regret upon the cash injection for updating the portfolio. The investor's objective is to minimise his expected total regret. Regarding the regret functions, they are closely related to utility functions used in utility maximisation problems in the above-mentioned studies. Similar to the definition of indifference prices based on utility optimisation problems, the regret indifference price of an option contract is defined as the price that allows the investor to enter the contract without increasing his expected total regret.

The regret optimisation problem in this work extends the utility maximisation problem in discrete-time financial market models by allowing the investor's preference to be different at different time steps. The most similar problem to the regret optimisation problem in this thesis is the asset-liability management problem which is studied by Pennanen (2014). He proves the existence of solution of his asset-liability management problem, and he also shows that the indifference prices of cash flows are within the no-arbitrage price interval. In this thesis we establish these two results in a different setting. In comparison with our study, although the optimisation problem in Pennanen (2014) allows convex transaction costs, the investor's liabilities are restricted to cash flows instead of portfolio flows. Additionally, numerical approaches for solving his optimisation problem and for computing the option prices have not been developed. In our work, we provide a numerical method to compute the investor's minimal regret and calculate option indifference prices. Moreover, we also provide substantial numerical results for the optimal cash injection strategy, and for the regret indifference prices of various European options in the models with large number of steps.

In this thesis we show that there exists a solution to our regret optimisation problem under the assumption of robust no-arbitrage. Moreover, we prove that the indifference prices of portfolio flows are within the no-arbitrage price interval. However, the calculation of indifference prices are challenging due to the difficulty of solving the regret optimisation problem. The main difficulty is that the cost function (which is used to compute the costs of creating a portfolio) is not differentiable at the origin when the transaction costs are non-zero. In order to calculate the indifference prices, we introduce a Lagrangian dual optimisation problem. It turns out that the solutions of this dual problem are very helpful for computing the indifference prices and solving the regret optimisation problem. Finally, under a sequence of exponential type regret functions, we find an algorithm to numerically compute the regret indifference prices. The numerical results show that the price interval based on regret indifference pricing can be much narrower than the price interval derived from superhedging. For the investors, such smaller price interval can be helpful for them to choose the "right" option prices.

This thesis is organised as follows.

In Chapter 2, we firstly introduce the discrete financial market model with proportional transaction costs. Then we provide a number of concepts such as solvency cones with their dual spaces, self-financing trading strategies, and no-arbitrage. In Theorem 2.6, we present the robust no-arbitrage condition introduced in Schachermayer (2004, Definition 1.9), and this condition is assumed to hold true throughout this thesis. Subsequently, we define the notion of a flow option (a sequence of European options with possibly different maturity dates) which extend the notion of a European option, and then define the superhedging prices of a flow option. In (2.26)-(2.27), we provide a link between the superhedging prices of flow options and that of European options.

The main contribution of this work is in Chapters 3-5. At the start of Chapter 3 we introduce the notion of a regret function and then introduce the regret minimisation problem (3.8). The problem (3.8) covers discrete-time versions of optimal investment and consumption problems and utility maximisation problems; see Examples 3.12 and 3.13. We show that the value function of the regret minimisation problem is lower semicontinuous, and that there exists a solution to the problem; see Theorem 3.15 and Corollary 3.16. After that, we reformulate the regret minimisation problem as a constrained optimisation problem in (3.19). Then a Lagrangian dual problem of (3.19) is defined in (3.35). The strong duality of the problems (3.19) and (3.35) is established in Theorem 3.31. Subsequently, we define the indifference prices of flow options. Moreover, Theorem 3.39 shows that the indifference prices are within the no-arbitrage price interval.

Chapter 4 provides a number of technical results used for the study in the next chapter. The results in this chapter does not rely on any financial market model or any result from previous chapters. Firstly, we introduce a minimisation problem for which the value function is formulated as an extended convex hull of a collection of convex functions. Then, in Theorem 4.3, we show that the value function is convex. Moreover, in Theorem 4.13, we establish the existence of a solution and the continuity of the value function. Subsequently, we present an example of the minimisation problem. In this example, we present a method to explicitly calculate the solutions to this problem by considering all different cases of the values of given parameters.

Chapter 5 concerns the dual optimisation problem (3.35) under a sequence of exponential regret functions. We first show that the solutions to this dual problem can be used to solve the problem (3.19); see Theorems 5.5 and 5.6. Moreover, based on the solutions to the dual problem, Theorem 5.7 provides formulae for computing the indifference prices of any flow option. By developing a dynamic programming algorithm, we can construct a solution to the dual problem (3.35); see Theorem 5.20. However, computing this solution is difficult. Thus, we propose a method to solve (3.35) numerically. Finally, in a binary market model, we produce a number of examples to compute the solution to (3.19) and the indifference prices of flow options with various payoffs.

Chapter 2

The model

2.1 The market model with proportional transaction costs

Consider a financial market model with discrete trading dates $t = 0, \dots, T$ and a finite probability space $(\Omega, \mathcal{F}, \mathbb{P})$ that is equipped with a filtration $(\mathcal{F}_t)_{t=0}^T$. Assume without loss of generality that $\mathcal{F}_0 = \{\Omega, \emptyset\}$, $\mathcal{F}_T = \mathcal{F} = 2^\Omega$ and $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$. For all $t = 0, \dots, T$, we denote by Ω_t the collection of atoms of $\mathcal{F}_t$. Moreover, the elements of Ω_t are called nodes of the model at time t.

For all $t = 0, \dots, T$ and $d \in \mathbb{N}$, let $\mathcal{L}_t^d$ be the space of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random variables, and define $\mathcal{L}_t = \mathcal{L}_t^1$. Moreover, let $\mathcal{L}_{t+}^d$ be the family of nonnegative random variables in $\mathcal{L}_t^d$. For any $x \in \mathcal{L}_t^d$, we have for all $\nu \in \Omega_t$ that $x(\omega) = x(\omega')$ for all $\omega, \omega' \in \nu$, and sometimes we use $x(\nu)$ to represent this common value. Let $\mathcal{N}^d$ be the space of adapted $\mathbb{R}^d$-valued processes, and define $\mathcal{N} = \mathcal{N}^1$.

The financial market model consists of two assets. Trading in the risky asset, *stock*, is subject to proportional transaction costs. At any time step $t = 0, \dots, T$, a share can be bought for the given *ask price* S_t^a and sold for the given *bid price* S_t^b, where $S_t^a \geq S_t^b > 0$. We assume that $S^a = (S_t^a)_{t=0}^T \in \mathcal{N}$ and $S^b = (S_t^b)_{t=0}^T \in \mathcal{N}$ respectively.

The risk-free asset, *cash*, is taken to be a risk-free bond with zero interest rate. Its price is constant and equal to 1 for all $t = 0, \dots, T$. Equivalently, asset prices in our market model can be considered as discounted prices.

For all $x \in \mathbb{R}$, let

$$x_+ := \max\{x, 0\}, \qquad x_- := -\min\{x, 0\}.$$

Fix any $t = 0, \dots, T$. The cost of setting up a portfolio $x = (x^b, x^s)$ at time t is

$$\phi_t(x) := x^b + x^s_+ S^a_t - x^s_- S^b_t$$

and the liquidation value of the portfolio x at time t is

$$x^b - x^s_- S^a_t + x^s_+ S^b_t = -\phi_t(-x).$$

Observe that $S^a_t, S^b_t \in \mathcal{L}_t$ and hence the function ϕ_t based on S^a_t and S^b_t is an $\mathcal{F}_t$*-measurable random function*. See Definition A.16 and comments following it for the definition of measurable random function and relevant properties and notation used in this work. The function ϕ_t is convex because $S^a_t \geq S^b_t$. In addition, we have

$$\phi_t(x) \geq x^b + x^s S \geq -\phi_t(-x) \text{ for all } S \text{ such that } S^b_t \leq S \leq S^a_t. \tag{2.1}$$

For every $\omega \in \Omega$, let

$$\begin{aligned} \mathcal{K}^\omega_t &:= \left\{ x \in \mathbb{R}^2 \middle| -\phi^\omega_t(-x) \geq 0 \right\} \\ &= \left\{ (x^b, x^s) \in \mathbb{R}^2 \middle| x^b - x^s_- S^a_t(\omega) + x^s_+ S^b_t(\omega) \geq 0 \right\} \\ &= \left\{ (x^b, x^s) \in \mathbb{R}^2 \middle| x^b + x^s S^b_t(\omega) \geq 0,\, x^b + x^s S^a_t(\omega) \geq 0 \right\}, \end{aligned} \tag{2.2}$$

which is the collection of portfolios with nonnegative liquidation value at time t and scenario ω. We shall refer $\mathcal{K}^\omega_t$ as the *solvency cone* at time t and scenario ω. Observe that $\mathcal{K}_t$ is determined by S^a_t and S^b_t and hence it is an $\mathcal{F}_t$*-measurable set-valued function*. See Definition A.12 and the comments following it for the definition of measurable set-valued function and relevant properties and notation used in this work. Note that, for each $\omega \in \Omega$, the set $\mathcal{K}^\omega_t \subseteq \mathbb{R}^2$ is a polyhedral cone and hence closed.

Remark 2.1. Fix any $\omega \in \Omega$. The graph of $\mathcal{K}^\omega_t$ is presented in Figure 2.1 for the case when $S^b_t(\omega) < S^a_t(\omega)$. In the case when $S^a_t(\omega) = S^b_t(\omega)$, the polyhedral cone $\mathcal{K}^\omega_t$ is a half space, and the vectors $(S^a_t(\omega), -1)$ and $(-S^b_t(\omega), 1)$ are on the same line. Thus, apart from $(S^a_t(\omega), -1)$, and $(-S^b_t(\omega), 1)$, at least one additional vector is needed to generate $\mathcal{K}^\omega_t$. The choice of additional vectors is not unique, and here we chose the vectors $(1, 0)$ and $(0, 1)$. From the graph, the polyhedral cone $\mathcal{K}^\omega_t$ is generated by $(1, 0)$, $(0, 1)$, $(S^a_t(\omega), -1)$, and $(-S^b_t(\omega), 1)$, in other words,

$$\mathcal{K}^\omega_t = \{\alpha(1,0) + \beta(0,1) + \gamma(S^a_t(\omega), -1) + \delta(-S^b_t(\omega), 1) \mid \alpha, \beta, \gamma, \delta \in [0, \infty)\}. \tag{2.3}$$

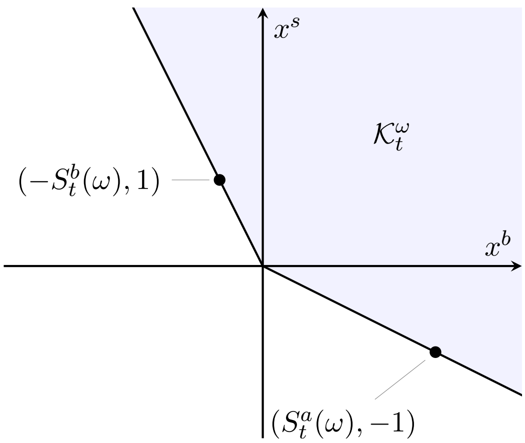


Figure 2.1: The solvency cone $\mathcal{K}_t^\omega$ at time t and scenario ω

For all $\omega \in \Omega$, the family $-\mathcal{K}_t^\omega$ can be presented as

$$\begin{aligned} -\mathcal{K}_t^\omega &= \left\{ x \in \mathbb{R}^2 \mid -\phi_t^\omega(x) \geq 0 \right\} \\ &= \left\{ x \in \mathbb{R}^2 \mid \phi_t^\omega(x) \leq 0 \right\} \end{aligned} \tag{2.4}$$

which is the collection of portfolios that can be created from zero cash at time t and scenario ω.

A *trading strategy* is a sequence $(y_t)_{t=0}^T \in \mathcal{N}^2$ of portfolios coupled with an initial endowment $y_{-1} \in \mathbb{R}^2$. The collection of trading strategies is denoted by $\mathcal{N}^{2\prime}$. For $t = 0, \ldots, T-1$, the portfolio y_t is held during the interval $(t, t+1]$ and y_T is the terminal portfolio.

Definition 2.2. We call a trading strategy $y = (y_t)_{t=-1}^T$ *self-financing* if

$$\Delta y_t := y_t - y_{t-1} \in -\mathcal{K}_t \text{ for all } t = 0, \ldots, T.$$

We denote the collection of self-financing strategies by Φ.

A trading strategy is self-financing if there is no injection of funds beyond the initial endowment, in other words, the change in the portfolio holdings Δy_t at each time step t can be created without additional investment.

Remark 2.3. The definition of the self-financing property above allows for the withdrawal of funds. For example, define $y \in \mathcal{N}^{2\prime}$ as

$$\begin{aligned} y_{-1} &:= (1, 1) \\ y_t &:= y_{t-1} - \tfrac{1}{T+1}(1, 1) \text{ for all } t = 0, \ldots, T. \end{aligned}$$

Then $\Delta y_t = -\frac{1}{T+1}(1,1) \in -\mathcal{K}_t$ for all t and hence $y \in \Phi$. By the construction of y, the portfolio $\frac{1}{T+1}(1,1)$ is withdrawn at each time t.

2.2 Arbitrage and dual spaces

In this section, we consider two different notions of absence of arbitrage. The first, *no-arbitrage,* is important in the arbitrage pricing theory which will be discussed in the next section. The second, *robust no-arbitrage*, will turn out to be a sufficient condition for the existence of a solution to the regret minimisation problem in the next chapter. Define

$$\bar{\mathcal{P}} := \left\{ (\mathbb{Q}, S) \middle| \, \mathbb{Q} \ll \mathbb{P},\ S \text{ a } \mathbb{Q}\text{-martingale},\ S^b_t \leq S_t \leq S^a_t \right\}, \tag{2.5}$$

$$\mathcal{P} := \left\{ (\mathbb{Q}, S) \middle| \, \mathbb{Q} \sim \mathbb{P},\ S \text{ a } \mathbb{Q}\text{-martingale},\ S^b_t \leq S_t \leq S^a_t \right\}, \tag{2.6}$$

where "$\mathbb{Q} \ll \mathbb{P}$" in (2.5) means that $\mathbb{Q}$ is a probability measure that is absolute continuous with respect to $\mathbb{P}$, and "$\mathbb{Q} \sim \mathbb{P}$" in (2.6) means that $\mathbb{Q}$ and $\mathbb{P}$ are equivalent. We shall refer the elements of $\bar{\mathcal{P}}$ ($\mathcal{P}$) as (equivalent) martingale pairs. Notice that $\mathcal{P} \subseteq \bar{\mathcal{P}}$.

Remark 2.4. If there are no transaction costs, then $S^b = S^a$ and for any $(\mathbb{Q}, S) \in \mathcal{P}$ the probability measure $\mathbb{Q}$ is an *equivalent martingale measure* in the friction-free model with stock price $S = S^b = S^a$.

We denote the collection of terminal portfolios associated with self-financing trading strategies with zero initial endowment by

$$\mathcal{A}_T := \left\{ y_T \middle| (y_t)_{t=-1}^T \in \Phi \text{ and } y_{-1} = 0 \right\}.$$

The following result is due to Kabanov & Stricker (2001, Theorem 1) (see also Schachermayer (2004, Theorem 1.7 and pp. 24-25)). We follow Schachermayer (2004) in referring to the condition in the following result as the no-arbitrage condition. Although formulated differently, this is equivalent to the notion of weak no-arbitrage introduced by Kabanov & Stricker (2001).

Theorem 2.5. *The market model satisfies the* no-arbitrage condition

$$\mathcal{A}_T \cap \mathcal{L}^2_{T+} = \{0\}$$

if and only if $\mathcal{P} \neq \emptyset$.

In the present setting, the *robust no-arbitrage condition* introduced in Schachermayer (2004, Definition 1.9) is satisfied if there exists $(\tilde{S}^b, \tilde{S}^a) \in \mathcal{N}^2$ such that the following two conditions are satisfied:

1. For every $t = 0, \dots, T$, we have $\tilde{S}^b_t \le \tilde{S}^a_t$ and $[\tilde{S}^b_t, \tilde{S}^a_t]$ is contained in the relative interior of $[S^b_t, S^a_t]$, in other words,

$$\begin{aligned} S^b_t = S^a_t \quad &\Longrightarrow \quad S^b_t = \tilde{S}^b_t = \tilde{S}^a_t = S^a_t, \\ S^b_t < S^a_t \quad &\Longrightarrow \quad S^b_t < \tilde{S}^b_t \le \tilde{S}^a_t < S^a_t. \end{aligned}$$

2. The market model with stock prices modelled by $(\tilde{S}^b, \tilde{S}^a)$ (instead of (S^b, S^a)) satisfies the no-arbitrage condition.

From Schachermayer (2004, Theorem 1.7 and pp. 24-25), we have following equivalent presentation of the robust no-arbitrage condition.

Theorem 2.6. *The market model satisfies the* robust no-arbitrage condition *if and only if there exists $(\mathbb{Q}, S) \in \mathcal{P}$ such that S_t is in the relative interior of $[S^b_t, S^a_t]$ for all $t = 0, \dots, T$.*

Observe that the robust no-arbitrage condition implies that $\mathcal{P} \neq \emptyset$ and hence implies that the no-arbitrage condition holds. In the remainder of our work, we will always assume that the robust no-arbitrage condition holds true.

Define

$$\Psi := \left\{ y \in \mathcal{N}^{2\prime} \middle| \, y_{-1} = y_T = 0 \right\}. \tag{2.7}$$

Then

$$\begin{aligned} \Phi \cap \Psi &= \left\{ y \in \mathcal{N}^{2\prime} \middle| \, y_{-1} = y_T = 0, \; \Delta y_t \in -\mathcal{K}_t \text{ for all } t = 0, \dots, T \right\} \\ &= \left\{ y \in \mathcal{N}^{2\prime} \middle| \, y_{-1} = y_T = 0, \; \phi_t(\Delta y_t) \le 0 \text{ for all } t = 0, \dots, T \right\} \end{aligned} \tag{2.8}$$

is the collection of self-financing trading strategies with both initial endowment and final value equal to zero. It turns out that the linearity of $\Phi \cap \Psi$ is crucial for the existence of a solution to the optimisation problem that will be studied in the next chapter. Note that, while robust no-arbitrage is assumed, the weaker no-arbitrage condition is sufficient for the following result to hold true.

Proposition 2.7. *If $y \in \Phi \cap \Psi$, then $\phi_t(\Delta y_t) = 0$ for all $t = 0, \dots, T$.*

Proof. Let $y \in \Phi \cap \Psi$, and suppose by contradiction that there exists some $t^* = 0, \dots, T$ such that

$$\mathbb{P}(\phi_{t^*}(\Delta y_{t^*}) < 0) > 0.$$

This means that there exists $\nu \in \Omega_{t^*}$ such that $\epsilon := \phi^\nu_{t^*}(\Delta y_{t^*}(\nu)) < 0$, with $\epsilon \in \mathbb{R}$ by the adaptedness of y and stock price processes; see the comments following Definition A.16 for $\phi_{t^*}^{\nu}$. Define $z = (z_t)_{t=-1}^{T} \in \mathcal{N}^{2\prime}$ as

$$z_t = \begin{cases} y_t - (\epsilon, 0) & \text{on } \nu \text{ if } t \geq t^*, \\ y_t & \text{otherwise.} \end{cases}$$

Then $z_{-1} = 0$ and

$$\Delta z_t = \begin{cases} \Delta y_t - (\epsilon, 0) \in -\mathcal{K}_t & \text{on } \nu \text{ if } t = t^*, \\ \Delta y_t \in -\mathcal{K}_t & \text{otherwise,} \end{cases}$$

which implies that $z \in \Phi$ and hence $z_T \in \mathcal{A}_T$. However $y_T = 0$ gives that

$$z_T = \begin{cases} y_T - (\epsilon, 0) = -(\epsilon, 0) & \text{on } \nu, \\ y_T = 0 & \text{on } \Omega \backslash \nu, \end{cases}$$

in other words, we have $z_T \in \mathcal{L}^2_{T+}$ with $z_T \neq 0$. Therefore, the no-arbitrage condition is violated and hence the result follows. □

The following result follows from Proposition 2.7, and it shows that robust no-arbitrage is sufficient for $\Phi \cap \Psi$ to be a *linear space;* for the definition of a linear space see Roman (2008, pp. 35-36).

Proposition 2.8. *The set $\Phi \cap \Psi$ is a linear space.*

Proof. Our main objective is to prove that for every $y \in \Phi \cap \Psi$ we have

$$\Delta y_t^s = 0 \text{ on } \{S_t^b < S_t^a\} \text{ for all } t = 0, \dots, T. \tag{2.9}$$

This is shown below. Taking (2.9) as given, fix any $y \in \Phi \cap \Psi$ and $t = 0, \dots, T$. Then (2.9) implies that

$$\phi_t(-\Delta y_t) = \phi_t(-(\Delta y_t^b, 0)) = -\phi_t((\Delta y_t^b, 0)) = -\phi_t(\Delta y_t) \text{ on } \{S_t^b < S_t^a\}.$$

Combining this with ϕ_t being a linear function on $\{S_t^b = S_t^a\}$, it follows that

$$\phi_t(-\Delta y_t) = -\phi_t(\Delta y_t) = 0$$

by Proposition 2.7. Thus $-y \in \Phi \cap \Psi$. Since $\Phi \cap \Psi$ is also a convex cone, it has to be a linear space.

Now, we are going to show that (2.9) holds true for all $y \in \Phi \cap \Psi$. First of all, fix any $(\mathbb{Q}, S) \in \mathcal{P}$ satisfying the conditions in Theorem 2.6. For any $y \in \Phi \cap \Psi$, from $y_T = (y_T^b, y_T^s) = 0$ and (2.1) together with Proposition 2.7, it follows that

$$y^b_{T-1} + y^s_{T-1} S_T = -\Delta y^b_T - \Delta y^s_T S_T \geq -\phi_T(\Delta y_T) = 0. \tag{2.10}$$

Since $\left(y^b_{t-1} + y^s_{t-1} S_t\right)_{t=0}^T$ is a $\mathbb{Q}$-supermartingale (Roux, Tokarz & Zastawniak 2008, Lemma 7.1), we have

$$y^b_{t-1} + y^s_{t-1} S_t \geq \mathbb{E}_{\mathbb{Q}}\left[y^b_{t'-1} + y^s_{t'-1} S_{t'} \middle| \mathcal{F}_t \right] \text{ for all } 0 \leq t \leq t' \leq T. \tag{2.11}$$

Combining (2.11) and (2.10), we have

$$y^b_{t-1} + y^s_{t-1} S_t \geq \mathbb{E}_{\mathbb{Q}}\left[y^b_{T-1} + y^s_{T-1} S_T \middle| \mathcal{F}_t \right] \geq 0 \text{ for all } t = 0, \ldots, T. \tag{2.12}$$

Suppose by contradiction that there exists some $t^* = 0, \ldots, T$ such that

$$\nu := \{\Delta y^s_{t^*} \neq 0\} \cap \{S^b_{t^*} < S^a_{t^*}\} \neq \emptyset.$$

Since $(\mathbb{Q}, S)$ satisfies the conditions in Theorem 2.6, we can present ν as

$$\nu = \{\Delta y^s_{t^*} \neq 0\} \cap \{S^b_{t^*} < S_{t^*} < S^a_{t^*}\}.$$

Combining this and Proposition 2.7 together with $y \in \Phi \cap \Psi$, it follows that

$$\Delta y^b_{t^*} + \Delta y^s_{t^*} S_{t^*} < \phi_{t^*}(\Delta y_{t^*}) = 0 \text{ on } \nu,$$

in other words,

$$y^b_{t^*-1} + y^s_{t^*-1} S_{t^*} > y^b_{t^*} + y^s_{t^*} S_{t^*} \text{ on } \nu. \tag{2.13}$$

Consider the following two cases. In the case when $t^* = T$, we have from (2.13) and $y_T = 0$ that

$$y^b_{T-1} + y^s_{T-1} S_T > 0 \text{ on } \nu.$$

In the case when $t^* < T$, it follows from (2.13) and the fact that S is a $\mathbb{Q}$-martingale that

$$y^b_{t^*-1} + y^s_{t^*-1} S_{t^*} > y^b_{t^*} + y^s_{t^*} \mathbb{E}_{\mathbb{Q}}\left[S_{t^*+1} \middle| \mathcal{F}_{t^*} \right] = \mathbb{E}_{\mathbb{Q}}\left[y^b_{t^*} + y^s_{t^*} S_{t^*+1} \middle| \mathcal{F}_{t^*} \right] \text{ on } \nu.$$

Combining this with $y^b_{t^*} + y^s_{t^*} S_{t^*+1} \geq 0$ by (2.12), it follows that

$$y^b_{t^*-1} + y^s_{t^*-1} S_{t^*} > 0 \text{ on } \nu.$$

Thus, we always have $y^b_{t^*-1} + y^s_{t^*-1} S_{t^*} > 0$ on ν. Combining this with (2.11), we have

$$0 < \mathbb{E}_{\mathbb{Q}}\left[y^b_{t^*-1} + y^s_{t^*-1}S_{t^*}\right] \leq y^b_{-1} + y^s_{-1}S_0.$$

This contradicts $y^b_{-1} + y^s_{-1}S_0 = 0$ because $y_{-1} = 0$. This establishes (2.9) and hence completes the proof. □

Let $\cdot$ denote the scalar product in $\mathbb{R}^d$. For any cone $C \subseteq \mathbb{R}^d$, we write the *polar* C^+ of $-C$ as

$$C^+ = \left\{ y \in \mathbb{R}^d \middle| y \cdot x \geq 0 \text{ for all } x \in C \right\}. \tag{2.14}$$

For all $t = 0, \ldots, T$, we define the set-valued function $\mathcal{K}^+_t$ as

$$\mathcal{K}^{+\omega}_t := \mathcal{K}^{\omega+}_t \text{ for all } \omega \in \Omega. \tag{2.15}$$

Notice that $\mathcal{K}^+_t$ is determined by S^a_t and S^b_t which means it is $\mathcal{F}_t$-measurable. Moreover, the following result provides an expression for $\mathcal{K}^+_t$.

Lemma 2.9. *For all $t = 0, \ldots, T$ and $\omega \in \Omega$, we have*

$$\mathcal{K}^{+\omega}_t = \left\{ (z^b, z^s) \in [0,\infty)^2 \middle| z^b S^b_t(\omega) \leq z^s \leq z^b S^a_t(\omega) \right\}.$$

Proof. Notice that, for any $(z^b, z^s) \in \mathbb{R}^2$, we have

$$\begin{aligned}
z^b \geq 0 &\iff (z^b, z^s) \cdot (1,0) \geq 0, \\
z^s \geq 0 &\iff (z^b, z^s) \cdot (0,1) \geq 0, \\
z^b S^a_t(\omega) \geq z^s &\iff (z^b, z^s) \cdot (S^a_t(\omega), -1) \geq 0, \\
z^s \geq z^b S^b_t(\omega) &\iff (z^b, z^s) \cdot (-S^b_t(\omega), 1) \geq 0.
\end{aligned}$$

This means

$$\begin{aligned}
&\left\{ (z^b, z^s) \in [0,\infty)^2 \middle| z^b S^b_t(\omega) \leq z^s \leq z^b S^a_t(\omega) \right\} \\
&\quad = \left\{ z \in \mathbb{R}^2 \middle| z \cdot y \geq 0 \text{ for all } y = (1,0), (0,1), (S^a_t(\omega), -1), (-S^b_t(\omega), 1) \right\}.
\end{aligned}$$

From (2.3), we have

$$\begin{aligned}
&\left\{ z \in \mathbb{R}^2 \middle| z \cdot x \geq 0 \text{ for all } x \in \mathcal{K}^\omega_t \right\} \\
&\quad \subseteq \left\{ z \in \mathbb{R}^2 \middle| z \cdot y \geq 0 \text{ for all } y = (1,0), (0,1), (S^a_t(\omega), -1), (-S^b_t(\omega), 1) \right\}.
\end{aligned}$$

The opposite set inclusion also holds. Indeed, fix any

$$z \in \left\{ z \in \mathbb{R}^2 \middle| z \cdot y \geq 0 \text{ for all } y = (1,0), (0,1), (S^a_t(\omega), -1), (-S^b_t(\omega), 1) \right\}.$$

Then we have for all $\alpha, \beta, \gamma, \delta \in [0, \infty)$ that

$$\begin{aligned} z \cdot &\Big[\alpha\,(1,0) + \beta\,(0,1) + \gamma(S_t^a(\omega), -1) + \delta(-S_t^b(\omega), 1)\Big] \\ &\qquad = \alpha z \cdot (1,0) + \beta z \cdot (0,1) + \gamma z \cdot (S_t^a(\omega), -1) + \delta z \cdot (-S_t^b(\omega), 1) \geq 0, \end{aligned}$$

and hence $z \cdot x \geq 0$ for all $x \in \mathcal{K}_t^\omega$ by (2.3). Thus, we can conclude that

$$\begin{aligned} &\Big\{ z \in \mathbb{R}^2 \Big| \, z \cdot x \geq 0 \text{ for all } x \in \mathcal{K}_t^\omega \Big\} \\ &\qquad\qquad = \Big\{ (z^b, z^s) \in [0,\infty)^2 \Big| \, z^b S_t^b(\omega) \leq z^s \leq z^b S_t^a(\omega) \Big\}. \end{aligned}$$

Combining this with (2.14) and (2.15), the result follows. □

Remark 2.10. Let $t = 0, \ldots, T$ and $z \in \mathcal{K}_t^+$. Combining Lemma 2.9 and $0 < S_t^b \leq S_t^a$, we have for all $\omega \in \Omega$ that either $z(\omega) = 0$ or $z(\omega) \in (0,\infty)^2$.

Define

$$\begin{aligned} \mathcal{C} &:= \Big\{ (z_t)_{t=0}^T \in \mathcal{N}^2 \Big| \, z \text{ a martingale, } z_t \in \mathcal{K}_t^+ \backslash \{0\} \text{ for all } t = 0, \ldots, T \Big\}, \\ \bar{\mathcal{C}} &:= \Big\{ (z_t)_{t=0}^T \in \mathcal{N}^2 \Big| \, z \text{ a martingale, } z_t \in \mathcal{K}_t^+ \text{ for all } t = 0, \ldots, T \Big\}. \end{aligned} \tag{2.16}$$

The elements of $\mathcal{C}$ are called *consistent pricing processes* which are introduced by Kabanov & Stricker (2001, p. 191) and Schachermayer (2004, Definition 1.5). The difference between $\mathcal{C}$ and $\bar{\mathcal{C}}$ is that the processes in $\bar{\mathcal{C}}$ are allowed to be zero at some time steps.

Remark 2.11. Observe that if $(z_t)_{t=0}^T \in \bar{\mathcal{C}}$, then the martingale property of $(z_t)_{t=0}^T$ implies that

$$\mathbb{E}\left[z_k \mid \mathcal{F}_t\right] = z_t = 0 \text{ on } \{z_t = 0\} \text{ for all } 0 \leq t \leq k \leq T,$$

Combining this with $z_k \in \mathcal{K}_k^+$ and Remark 2.10, it follows that

$$z_k = 0 \text{ on } \{z_t = 0\} \text{ for all } 0 \leq t \leq k \leq T.$$

The relationship between $\mathcal{C}$ and $\mathcal{P}$ is one-to-one up to a nonnegative factor; see Schachermayer (2004, pp. 24-25). In addition, Lemma 2.13 below implies that the relationship between $\bar{\mathcal{P}}$ and $\bar{\mathcal{C}}$ is also one-to-one up to a nonnegative factor.

For convenience, for every probability measure $\mathbb{Q}$ on $\mathcal{F}$ satisfying $\mathbb{Q} \ll \mathbb{P}$, we write

$$\Lambda_t^{\mathbb{Q}} := \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t\right] \text{ for all } t = 0, \ldots, T, \tag{2.17}$$

where $\frac{d\mathbb{Q}}{d\mathbb{P}}$ is the Radon-Nikodym density of $\mathbb{Q}$ with respect to $\mathbb{P}$. For any $t = 0, \ldots, T$ and $\nu \in \Omega_t$, the value $\Lambda_t^{\mathbb{Q}}(\nu)$ can be presented as

$$\Lambda_t^{\mathbb{Q}}(\nu) = \mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \middle| \mathcal{F}_t \right](\nu) = \frac{1}{\mathbb{P}(\nu)} \sum_{\omega \in \nu} \mathbb{P}(\omega) \frac{\mathbb{Q}(\omega)}{\mathbb{P}(\omega)} = \frac{\mathbb{Q}(\nu)}{\mathbb{P}(\nu)}.$$

In particular, we have $\Lambda_0^{\mathbb{Q}} = 1$, and hence

$$\mathbb{E}\left[\Lambda_0^{\mathbb{Q}} \ln \Lambda_0^{\mathbb{Q}}\right] = \mathbb{E}\left[1 \ln 1\right] = 0. \tag{2.18}$$

Remark 2.12. Notice that the process $(\Lambda_t^{\mathbb{Q}})_{t=0}^T$ is a $\mathbb{P}$-martingale. Then for any $t = 0, \ldots, T$ and $x \in \mathcal{L}_t$, the expectation $\mathbb{E}_{\mathbb{Q}}[x]$ can be written as

$$\mathbb{E}_{\mathbb{Q}}[x] = \mathbb{E}\left[\Lambda_T^{\mathbb{Q}} x\right] = \mathbb{E}\left[\mathbb{E}\left[\Lambda_T^{\mathbb{Q}} \middle| \mathcal{F}_t\right] x\right] = \mathbb{E}\left[\Lambda_t^{\mathbb{Q}} x\right].$$

Define an indicator function

$$\mathbf{1}_A(x) := \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases} \tag{2.19}$$

for any set A.

Lemma 2.13. *The family $\bar{\mathcal{C}}$ defined in* (2.16) *can be presented as*

$$\bar{\mathcal{C}} = \left\{ \left(\lambda (1, S_t) \Lambda_t^{\mathbb{Q}}\right)_{t=0}^T \middle| \lambda \geq 0,\ (\mathbb{Q}, S) \in \bar{\mathcal{P}} \right\}.$$

Proof. Firstly, fix any $\lambda \geq 0$ and $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$. Define

$$z = (z_t)_{t=0}^T = (z_t^b, z_t^s)_{t=0}^T \in \mathcal{N}^2$$

as

$$z_t := \lambda(1, S_t)\Lambda_t^{\mathbb{Q}} \text{ for all } t = 0, \ldots, T. \tag{2.20}$$

Fix any $t = 0, \ldots, T$. From (2.20), we have on $\{z_t \neq 0\}$ that $z_t^b > 0$ and $\frac{z_t^s}{z_t^b} = S_t$, and hence

$$S_t^b \leq \frac{z_t^s}{z_t^b} \leq S_t^a.$$

Moreover, we have on $\{z_t = 0\}$ that $z_t^b = z_t^s = 0$. Then $z_t \in \mathcal{K}_t^+$ by Lemma 2.9. In addition, for each $t = 0, \ldots, T-1$, the definition of z_{t+1} gives

$$\mathbb{E}\left[z_{t+1} \mid \mathcal{F}_t\right] = \lambda \mathbb{E}\left[(1, S_{t+1})\Lambda_{t+1}^{\mathbb{Q}} \middle| \mathcal{F}_t\right].$$

Then it follows from Bayes' formula (Shreve 2004, Lemma 5.2.2) that

$$\begin{aligned}\mathbb{E}\left[z_{t+1} \mid \mathcal{F}_t\right] &= \lambda \mathbb{E}_{\mathbb{Q}}\left[(1, S_{t+1}) \middle| \mathcal{F}_t\right] \Lambda_t^{\mathbb{Q}} \\ &= \lambda \left(1, S_t\right) \Lambda_t^{\mathbb{Q}} = z_t.\end{aligned}$$

Thus, the process z is a martingale, and therefore $z \in \bar{\mathcal{C}}$.

Now, fix any $z = (z^b, z^s) \in \bar{\mathcal{C}}$. If $z_0 = 0$, then it follows from Remark 2.11 that

$$z_t = 0 = 0(1, S_t)\Lambda_t^{\mathbb{Q}} \text{ for all } t = 0, \ldots, T$$

for any choice of $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$. Suppose $z_0 \neq 0$; then $z_0^b > 0$ by Remark 2.10. Define $\lambda := z_0^b$ and a measure $\mathbb{Q}$ by means of its Radon-Nikodym density

$$\frac{d\mathbb{Q}}{d\mathbb{P}} := \frac{z_T^b}{z_0^b}.$$

Then $\mathbb{Q}\left(\Omega\right) = \frac{\mathbb{E}(z_T^b)}{z_0^b} = 1$ because z^b is a martingale, and hence $\mathbb{Q}$ is a probability measure. Moreover, we have

$$\Lambda_t^{\mathbb{Q}} = \mathbb{E}\left[\left.\frac{z_T^b}{z_0^b}\right| \mathcal{F}_t\right] = \frac{z_t^b}{z_0^b} \text{ for all } t = 0, \ldots, T. \tag{2.21}$$

Let us now define $S = (S_t)_{t=0}^T \in \mathcal{N}$. For every $t = 0, \ldots, T$, let

$$\nu_t := \{z_t^b = 0\}$$

and

$$S_t := \begin{cases} \frac{z_t^s}{z_t^b} & \text{on } \Omega \backslash \nu_t \\ S_t^b & \text{on } \nu_t. \end{cases}$$

Combining Remark 2.10 and (2.21), we have on ν_t that $z_t = 0 = \lambda(1, S_t)\Lambda_t^{\mathbb{Q}}$. Moreover, we have on $\Omega \backslash \nu_t$ that $z_t^b > 0$ and

$$z_t = \left(z_t^b, z_t^s\right) = z_0^b \left(1, \frac{z_t^s}{z_t^b}\right) \frac{z_t^b}{z_0^b} = \lambda \left(1, S_t\right) \Lambda_t^{\mathbb{Q}},$$

again by (2.21). It remains to show that $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$. Observe first that, for all $t = 0, \ldots, T-1$, the probability $\mathbb{Q}(\nu_{t+1})$ is

$$\mathbb{Q}\left(\nu_{t+1}\right) = \mathbb{E}_{\mathbb{Q}}\left[\mathbf{1}_{\nu_{t+1}}\right] = \mathbb{E}\left[\Lambda_{t+1}^{\mathbb{Q}} \mathbf{1}_{\nu_{t+1}}\right] = 0$$

by Remark 2.12 and (2.21). In addition, it follows from Remark 2.11 that

$\nu_{t+1} \supseteq \nu_t$, and hence we have on $\Omega \backslash \nu_{t+1}$ that

$$\begin{aligned} S_t &= \frac{z_t^s}{z_t^b} = \frac{z_0^b}{z_t^b} \frac{z_t^s}{z_0^b} \\ &= \frac{1}{\Lambda_t^{\mathbb{Q}}} \mathbb{E}\left[\left. \frac{z_{t+1}^s}{z_0^b} \right| \mathcal{F}_t \right] \\ &= \frac{1}{\Lambda_t^{\mathbb{Q}}} \mathbb{E}\left[\left. \frac{z_{t+1}^s}{z_{t+1}^b} \frac{z_{t+1}^b}{z_0^b} \right| \mathcal{F}_t \right] \\ &= \frac{1}{\Lambda_t^{\mathbb{Q}}} \mathbb{E}\left[\left. S_{t+1} \Lambda_{t+1}^{\mathbb{Q}} \right| \mathcal{F}_t \right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[S_{t+1} \mid \mathcal{F}_t \right] \end{aligned}$$

(Shreve 2004, Lemma 5.2.2). Thus, the process S is a $\mathbb{Q}$-martingale. Moreover, the definition of S together with Lemma 2.9 implies that

$$S_t^b \leq S_t \leq S_t^a \text{ for all } t = 0, \ldots, T,$$

which establishes $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$. □

2.3 Superhedging

A European option with payoff $c = (c^b, c^s) \in \mathcal{L}_T^2$ and physical delivery at maturity date T is a contract whereby the seller delivers the portfolio c at time T to the buyer. We call $y = (y_T)_{t=-1}^T \in \mathcal{N}^{2\prime}$ a *superhedging strategy for the seller* of the option c if $y \in \Phi$ and $y_T = c$. The lowest initial cash holding y_{-1}^b such that the seller is able to superhedge c with y starting from the initial endowment $(y_{-1}^b, 0)$ defines the *seller's superhedging price* of c, namely

$$\pi_{\mathrm{E}}^a(c) := \inf \left\{ y_{-1}^b \middle| y \in \Phi,\, y_{-1}^s = 0,\, y_T = c \right\}.$$

Theorem 2.14 below shows that the above infimum is attained. The quantity $\pi_{\mathrm{E}}^a(c)$ is the smallest amount of cash that enables the seller to meet his obligation of delivering c to the buyer without risk.

The buyer's superhedging price of the option c is defined in a similar manner. We call $y = (y_t)_{t=-1}^T$ a *superhedging strategy for the buyer* if $y \in \Phi$ and $y_T = -c$. The *buyer's superhedging price* of c is defined as

$$\pi_{\mathrm{E}}^b(c) := -\inf \left\{ y_{-1}^b \middle| y \in \Phi,\, y_{-1}^s = 0,\, y_T = -c \right\}.$$

Notice that

$$\pi_{\mathrm{E}}^b(c) = -\pi_{\mathrm{E}}^a(-c). \tag{2.22}$$

Again Theorem 2.14 below shows that the above infimum is attained. The quantity $\pi_{\mathrm{E}}^{b}(c)$ is the largest amount of cash that can be raised at time 0 by the buyer with the guarantee c at time T. Note that there is a symmetry between the seller's and buyer's position. The buyer, as the option holder, will receive c at time T. Equivalently, the buyer has to deliver $-c$ at time T. To fully cover this risk, the buyer has to receive at least $-\pi_{\mathrm{E}}^{b}(c)$ in cash from the seller, namely, pay at most $\pi_{\mathrm{E}}^{b}(c)$ to the seller.

Theorem 2.14. *(Roux & Zastawniak 2016, Theorems 4.4, 4.10). Suppose that $c = (c^b, c^s) \in \mathcal{L}_T^2$ is the payoff of a European option with maturity date T. Then the seller's and buyer's superhedging prices can be represented as*

$$\pi_{\mathrm{E}}^{a}(c) = \max_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}}\left[c^b + S_T c^s\right] = \sup_{(\mathbb{Q},S)\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[c^b + S_T c^s\right]$$

and

$$\pi_{\mathrm{E}}^{b}(c) = \min_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}}\left[c^b + S_T c^s\right] = \inf_{(\mathbb{Q},S)\in\mathcal{P}} \mathbb{E}_{\mathbb{Q}}\left[c^b + S_T c^s\right].$$

Moreover, if $x \geq \pi_{\mathrm{E}}^{a}(c)$, then there exists a superhedging strategy with initial endowment $(x, 0)$ for the seller of c, and if $x \leq \pi_{\mathrm{E}}^{b}(c)$, then there exists a superhedging strategy with initial endowment $(-x, 0)$ for the buyer of c.

The following result follows from Theorem 2.14.

Corollary 2.15. *Given a European option with payoff $c = (c^b, c^s) \in \mathcal{L}_T^2$ at maturity date T, we have*

$$\mathbb{E}_{\mathbb{Q}}\left[c^b + S_T c^s\right] \leq 0 \text{ for all } (\mathbb{Q}, S) \in \bar{\mathcal{P}}$$

if and only if there exists a superhedging strategy with zero initial endowment for the seller of c.

Proof. Note that $\mathbb{E}_{\mathbb{Q}}\left[c^b + S_T c^s\right] \leq 0$ for all $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$ if and only if

$$0 \geq \max_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}}\left[c^b + S_T c^s\right] = \pi_{\mathrm{E}}^{a}(c) \tag{2.23}$$

(Theorem 2.14). Since $0 \geq \pi_{\mathrm{E}}^{a}(c)$ if and only if there exists a superhedging strategy with zero initial endowment for the seller of c, the result follows. □

A *flow option* with payoff $c = (c_t)_{t=0}^{T} \in \mathcal{N}^2$ is a contract whereby the seller delivers the portfolio c_t at time t to the buyer for every $t = 0, \ldots, T$. This option can be seen as a portfolio of $T+1$ European options with maturity dates $0, \ldots, T$ and payoffs $c_0, \ldots, c_T$. We call $y = (y_t)_{t=-1}^{T} \in \mathcal{N}^{2\prime}$ a *superhedging* *strategy for the seller* of the flow option c if

$$\Delta y_t + c_t \in -\mathcal{K}_t \text{ for all } t = 0, \ldots, T \text{ and } y_T = 0.$$

Note that the superhedging strategy defined above does not have to be self-financing. Suppose $y = (y_t)_{t=-1}^T$ is a superhedging strategy for the seller of the flow option $(c_t)_{t=0}^T$. Then, at each time step $t = 0, \ldots, T$, the seller can create the portfolio $\Delta y_t + c_t$ from zero cash to meet his obligation by delivering c_t and manage his trading strategy by adding Δy_t to his current portfolio y_{t-1}. Thus the superhedging strategy y enables the seller to meet his obligation of delivering c to the buyer without risk. The lowest initial cash holding y_{-1}^b such that the seller is able to superhedge c with y starting from initial endowment $(y_{-1}^b, 0)$ defines the *seller's superhedging price* of c, namely

$$\pi_{\mathrm{F}}^a(c) := \inf\left\{ y_{-1}^b \,\middle|\, y \in \mathcal{N}^{2\prime} \text{ superhedges } c \text{ for the seller}, y_{-1}^s = 0 \right\}. \tag{2.24}$$

The buyer's superhedging price of the option c is defined in a similar manner. We call $y \in \mathcal{N}^{2\prime}$ a *superhedging strategy for the buyer* of c if

$$\Delta y_t - c_t \in -\mathcal{K}_t \text{ for all } t = 0, \ldots, T \text{ and } y_T = 0.$$

The *buyer's superhedging price* of c is defined as

$$\begin{aligned}
\pi_{\mathrm{F}}^b(c) &:= -\inf\left\{ y_{-1}^b \,\middle|\, y \in \mathcal{N}^{2\prime} \text{ superhedges } c \text{ for the buyer}, y_{-1}^s = 0 \right\} \\
&= -\inf\left\{ y_{-1}^b \,\middle|\, y \in \mathcal{N}^{2\prime} \text{ superhedges } -c \text{ for the seller}, y_{-1}^s = 0 \right\} \\
&= -\pi_{\mathrm{F}}^a(-c).
\end{aligned} \tag{2.25}$$

The following result gives a link between a superhedging strategy for the seller of the flow option c and a superhedging strategy for the seller of the European option with payoff $\sum_{t=0}^T c_t$ at maturity date T.

Lemma 2.16. *Let $c = (c_t)_{t=0}^T \in \mathcal{N}^2$ be a flow option.*

1. *If $(y_t)_{t=-1}^T \in \mathcal{N}^{2\prime}$ is a superhedging strategy for the seller of c, then $(x_t)_{t=-1}^T \in \mathcal{N}^{2\prime}$ defined by*

   $$x_{-1} := y_{-1}, \quad x_t := y_t + \textstyle\sum_{k=0}^t c_k \text{ for all } t = 0, \ldots, T$$

   is a superhedging strategy for the seller of the European option with payoff $\sum_{t=0}^T c_t$ at maturity date T.

2. *If $(y_t)_{t=-1}^T \in \mathcal{N}^{2\prime}$ is a superhedging strategy for the seller of the European*

option with payoff $\sum_{t=0}^T c_t$ *at maturity date* T*, then* $(x_t)_{t=-1}^T \in \mathcal{N}^{2\prime}$ *defined by*

$$x_{-1} := y_{-1}, \quad x_t := y_t - \textstyle\sum_{k=0}^t c_k \text{ for all } t = 0, \ldots, T$$

is a superhedging strategy for the seller of c*.*

Proof. In the first claim, it follows from $y_T = 0$ that $x_T = \sum_{t=0}^T c_t$. Moreover, we have

$$\Delta x_t = \Delta y_t + c_t \in -\mathcal{K}_t \text{ for all } t = 0, \ldots, T,$$

which implies that $(x_t)_{t=-1}^T \in \Phi$. Thus, the first claim holds true.

In the second claim, we have

$$\Delta x_t + c_t = \Delta y_t \in -\mathcal{K}_t \text{ for all } t = 0, \ldots, T,$$

and moreover $x_T = 0$ by $y_T = \sum_{t=0}^T c_t$. Thus, the second claim holds true. □

Let $z \in \mathbb{R}^2$ and let $c = (c_t)_{t=0}^T \in \mathcal{N}^2$ be a flow option. Lemma 2.16 implies that the following two statements are equivalent:

1. There exists a superhedging strategy $(y_t)_{t=-1}^T$ with initial endowment $y_{-1} = z$ for the seller of the flow option c;
2. There exists a superhedging strategy $(y_t)_{t=-1}^T$ with initial endowment $y_{-1} = z$ for the seller of the European option with payoff $\sum_{t=0}^T c_t$ at maturity date T.

This implies that

$$\pi_{\mathrm{F}}^a(c) = \pi_{\mathrm{E}}^a\left(\textstyle\sum_{t=0}^T c_t\right) \tag{2.26}$$

(cf. Corollary 3.32 of Tien (2011)). Moreover, we have

$$\pi_{\mathrm{F}}^b(c) = -\pi_{\mathrm{F}}^a(-c) = -\pi_{\mathrm{E}}^a\left(-\textstyle\sum_{t=0}^T c_t\right) = \pi_{\mathrm{E}}^b\left(\textstyle\sum_{t=0}^T c_t\right) \tag{2.27}$$

by (2.22).

Corollary 2.17. *Given a flow option* $c = (c_t)_{t=0}^T = (c_t^b, c_t^s)_{t=0}^T \in \mathcal{N}^2$*, we have*

$$\mathbb{E}_{\mathbb{Q}}\left[\textstyle\sum_{t=0}^T c_t^b + S_T \sum_{t=0}^T c_t^s\right] \le 0 \text{ for all } (\mathbb{Q}, S) \in \bar{\mathcal{P}} \tag{2.28}$$

if and only if there exists a superhedging strategy with zero initial endowment for the seller of c*.*

Proof. Corollary 2.15 implies that (2.28) holds true if and only if there exists a superhedging strategy with zero initial endowment for the seller of the

European option with payoff $\sum_{t=0}^{T} c_t$ at maturity date T. Then the result follows from the comments preceding (2.26). $\square$

Chapter 3

Regret optimisation

In this chapter, we will study an optimisation problem in the market model presented in Chapter 2 with the robust no-arbitrage condition being assumed. In this problem, we consider an investor who faces the liability of delivering a sequence of portfolios. At each time step, he also manages his financial position in the underlying assets. Moreover, his trading strategy is not required to be self-financing, in other words, he is allowed to inject extra cash beyond the initial endowment. The investor's regret upon cash injection for updating the portfolio in each trade is measured by a regret function, and his objective is to minimise his expected total regret. Section 3.1 introduces the notion of a regret function. Then Section 3.2 formulates the regret optimisation problem, and Section 3.2.1 studies the existence of a solution of this problem. In Section 3.2.2, the regret optimisation problem in Section 3.2 is reformulated to reduce the dimensionality of the control variable. Then, the Lagrangian dual problem of this reformulated optimisation problem is introduced in the Section 3.3. This dual problem will be used in Chapter 5 to study the algorithm of solving the regret optimisation problem numerically. Finally, based on the regret optimisation problem, Section 3.5 introduces a pricing method: regret indifferent pricing. The option prices derived from this pricing method will depend on the risk preference of the investor.

3.1 Regret function

This section introduces the notion of a *regret function*. Recall from the above introduction that the investor's trading strategy is not required to be self-financing and his regret/disutility of cash injection for updating the portfolio in each trade is measured by a regret function. If this injection is negative, then it is a consumption. First of all, the regret functions is an $\mathbb{R}\cup\{\infty\}$-valued function on $\mathbb{R}$. Moreover, it is natural to assume that the investor's regret of zero injection is zero. The investor prefers to inject less rather than more, so the regret function is required to be nondecreasing. In addition, the investor is assumed to be *risk averse* in the sense that the regret function is convex; the definition of convex function can be found in Appendix A.1.

Remark 3.1. The convexity of the regret function allows the investor to choose an injection strategy with lower "risks". For example, let $C \in \mathbb{R}$ and $X \in \mathcal{L}_T$ such that $C = \mathbb{E}[X]$. Consider the situation when the investor has to choose between injecting the constant amount of cash C and injecting the amount of cash X with possibly different outcomes. Suppose that the investor use a regret function v to compute his regret, and that he will choose the injection strategy (C or X) with lower expected regret. Then the right decision for the investor is to inject C because $v(C) = v(\mathbb{E}[X]) \leq \mathbb{E}[v(X)]$ (Jensen's inequality).

Define an indicator function

$$\delta_A(x) := \begin{cases} 0 & \text{if } x \in A \\ \infty & \text{otherwise} \end{cases} \tag{3.1}$$

for any set A; this function is different from the indicator function defined in (2.19).

Definition 3.2. We call $v : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ a *regret function* if

1. v is nondecreasing and convex on $\mathbb{R}$, and $v(0) = 0$;
2. v is lower semicontinuous, bounded from below and its recession function is $v^\infty = \delta_{(-\infty,0]}$.

The definitions of lower semicontinuity and recession function can be found in Appendix A.1. The properties listed under the second item are technical. We shall use them to prove that there exists a solution to the regret optimisation problem (3.8) in the next section. We denote the collection of all regret functions by $\mathbb{V}$.

Remark 3.3. Regret functions are closely related to *utility functions*. For example, if v is a regret function that is continuous, strictly increasing and strictly convex, then $U(x) = -v(-x)$ is a utility function in the sense of Definition 2.35 of Föllmer & Schied (2011).

Example 3.4. We have the following examples of regret functions.

1. The *exponential regret function*:

$$v(x) = e^{\alpha x} - 1 \text{ for all } x \in \mathbb{R},$$

   where $\alpha > 0$. The exponential regret function can be used when it is allowed to inject/withdraw arbitrarily large amount of cash. The parameter α describes the investor's risk aversion. The higher the value of α, the greater the risk aversion of the investor.

2. The *power regret function*:

$$v(x) = \begin{cases} \frac{b^\eta}{\eta} - \frac{1}{\eta}(b-x)^\eta & \text{if } x < b, \\ \infty & \text{if } x \geq b, \end{cases}$$

   where $b > 0$ and $\eta < 0$. The investor's cash injection is not allowed to be equal to or greater than b.

3. The regret function

$$v(x) = \delta_{(-\infty,0]}(x) \text{ for all } x \in \mathbb{R}$$

   is useful for the investor who does not wish to inject, but his regret is indifferent with respect to the size of withdrawals.

Fix any $v \in \mathbb{V}$ for the remainder of this section. We define

$$v^*(x) := \sup\{xy - v(y) | y \in \mathbb{R}\} \text{ for all } x \in \mathbb{R}. \tag{3.2}$$

Observe that v^* is the *conjugate function* (Rockafellar 1974, (3.10)) of v, and v^* will be used in the study of the dual optimisation problem in Section 3.3. We have $v^*(x) = \infty$ for any $x < 0$ because v is nondecreasing and $v(0) = 0$.

Remark 3.5. Fix any $x \geq 0$. Observe from (3.2) that

$$v^*(x) \geq xy - v(y) \text{ for all } y \in \mathbb{R}.$$

In particular, we have

$$v^*(x) \geq x \times 0 - v(0) = 0 \tag{3.3}$$

because $v(0) = 0$.

The following result implies that $v^*(x) < \infty$ for all $x \geq 0$. Combining this with (3.3), we have

$$v^*(x) \in [0, \infty) \text{ for all } x \geq 0. \tag{3.4}$$

This result also shows that the supremum in (3.2) is attained for all $x > 0$.

Proposition 3.6. *For all $x \geq 0$, we have $v^*(x) < \infty$. Moreover, in the situation when $x > 0$, there exists $\hat{y} \in \mathbb{R}$ such that*

$$x\hat{y} - v(\hat{y}) = v^*(x).$$

Proof. Fix any $x \geq 0$. Firstly, we are going to show that $v^*(x) < \infty$ by considering the following two cases. In the case when $x = 0$, we have

$$v^*(0) = \sup_{y\in\mathbb{R}} [-v(y)] = -\inf_{y\in\mathbb{R}} v(y) < \infty$$

since v is bounded from below.

In the case when $x > 0$, for convenience, we write

$$f_x(y) := v(y) - xy \text{ for all } y \in \mathbb{R}. \tag{3.5}$$

Observe that the function f_x is $\mathbb{R} \cup \{\infty\}$-valued, proper, closed, and convex. In addition, we have

$$-\inf_{y\in\mathbb{R}} f_x(y) = -\inf_{y\in\mathbb{R}} [v(y) - xy] = \sup_{y\in\mathbb{R}} [xy - v(y)] = v^*(x). \tag{3.6}$$

Since $v^\infty = \delta_{(-\infty,0]}$ and the recession function of the linear function $y \mapsto -xy$ is equal to itself (see Example A.4.1), the recession function f_x^∞ of f_x is

$$f_x^\infty(y) = \delta_{(-\infty,0]}(y) - xy$$

(Rockafellar 1997, Theorem 9.3). Notice that $f_x^\infty(1) = \infty > 1$ which means

$$(1, 1) \notin \operatorname{epi} f_x^\infty.$$

Moreover, it follows from $f_x^\infty(-1) = x > \frac{x}{2}$ that

$$(-1, \tfrac{x}{2}) \notin \operatorname{epi} f_x^\infty.$$

Thus, Lemma A.5 implies that f_x attains its infimum, in other words, there exists $\hat{y} \in \mathbb{R}$ such that

$$f_x(\hat{y}) = \inf_{y\in\mathbb{R}} f_x(y). \tag{3.7}$$

Therefore, it follows that

$$v^*(x) = -\inf_{y\in\mathbb{R}} f_x(y) = -f_x(\hat{y}) < \infty,$$

and this establishes that $v^*(x) < \infty$ for all $x \geq 0$. Combining (3.5), (3.7) and (3.6), it follows that

$$x\hat{y} - v(\hat{y}) = -f_x(\hat{y}) = -\inf_{y\in\mathbb{R}} f_x(y) = v^*(x).$$

This completes the proof. $\square$

The following example presents the conjugate function of each regret function introduced in Example 3.4. In particular, Example 3.4.3 implies that the condition $x > 0$ in Proposition 3.6 that guarantees the existence of $\hat{y} \in \mathbb{R}$ such that $x\hat{y} - v(\hat{y}) = v^*(x)$ is sufficient but not necessary.

Example 3.7. This example provides the values of v^* for each regret function v defined in Example 3.4. Notice from the comments following (3.2) that we always have $v^* = \infty$ on $(-\infty, 0)$. Thus, it is enough to compute the values of v^* on $[0, \infty)$.

1. Let $v\,(y) := e^{\alpha y} - 1$ for all $y \in \mathbb{R}$ where $\alpha > 0$. Notice that

$$v^*(0) = \sup_{y\in\mathbb{R}} [-v(y)] = 1 > -v(y') \text{ for all } y' \in \mathbb{R}.$$

For all $x > 0$, we have

$$\frac{d}{dy}\,[xy - v(y)] = x - \alpha e^{\alpha y} \text{ for all } y \in \mathbb{R}.$$

Then $y \mapsto xy - v(y)$ is continuous, and it is increasing on $(-\infty, \frac{1}{\alpha}\ln\frac{x}{\alpha}]$ and decreasing on $[\frac{1}{\alpha}\ln\frac{x}{\alpha}, \infty)$. This implies that $\hat{y} := \frac{1}{\alpha}\ln\frac{x}{\alpha}$ is the unique value in $\mathbb{R}$ that maximise $xy - v(y)$ over all $y \in \mathbb{R}$, and hence

$$x\hat{y} - v(\hat{y}) = \sup_{y\in\mathbb{R}} [xy - v(y)] = v^*(x).$$

By substituting $\hat{y} = \frac{1}{\alpha}\ln\frac{x}{\alpha}$ into $x\hat{y} - v(\hat{y})$, it yields $v^*(x) = \frac{x}{\alpha}\ln\frac{x}{\alpha} - \frac{x}{\alpha} + 1$. Combining this with $v^*(0) = 1$, we can conclude that

$$v^*(x) = \frac{x}{\alpha}\ln\frac{x}{\alpha} - \frac{x}{\alpha} + 1 \text{ for all } x \geq 0;$$

we always assume that $0\ln 0 = 0$ in this thesis.

2. Let $b > 0$ and $\eta < 0$. For any $y \in \mathbb{R}$, let

$$v\,(y) := \begin{cases} \frac{b^\eta}{\eta} - \frac{1}{\eta}\,(b-y)^\eta & \text{if } y < b, \\ \infty & \text{if } y \geq b. \end{cases}$$

Then

$$v^*(0) = \sup_{y\in\mathbb{R}} [-v(y)] = -\frac{b^\eta}{\eta} > -v(y') \text{ for all } y' \in \mathbb{R}.$$

Fix any $x > 0$. We have for any $y < b$ that

$$xy - v(y) = xy - \frac{b^\eta}{\eta} + \frac{1}{\eta}(b-y)^\eta$$

and

$$\frac{d}{dy}[xy - v(y)] = x - (b-y)^{\eta-1}.$$

Then $y \mapsto xy - v(y)$ is continuous on $(-\infty, b)$, and it is increasing on $(-\infty, b - x^{\frac{1}{\eta-1}}]$ and decreasing on $[b - x^{\frac{1}{\eta-1}}, b)$. Combining this with $xy - v(y) = -\infty$ for all $y \geq b$, it follows that $\hat{y} := b - x^{\frac{1}{\eta-1}}$ is the unique value in $\mathbb{R}$ that maximise $xy - v(y)$ over $y \in \mathbb{R}$. This implies

$$x\hat{y} - v(\hat{y}) = \sup_{y\in\mathbb{R}} [xy - v(y)] = v^*(x).$$

Then $v^*(x) = bx - \frac{\eta-1}{\eta}x^{\frac{\eta}{\eta-1}} - \frac{b^\eta}{\eta}$ by substituting $\hat{y} = b - x^{\frac{1}{\eta-1}}$ into $x\hat{y} - v(\hat{y})$. Combining this with $v^*(0) = -\frac{b^\eta}{\eta}$, it follows that

$$v^*(x) = bx - \frac{\eta-1}{\eta}x^{\frac{\eta}{\eta-1}} - \frac{b^\eta}{\eta} \text{ for all } x \geq 0.$$

3. Let $v(y) = \delta_{(-\infty,0]}(y)$ for all $y \in \mathbb{R}$. Observe that $\delta_{(-\infty,0]}(y) = \infty$ for $y > 0$. Then

$$v^*(0) = \sup_{y\in\mathbb{R}} \left[-\delta_{(-\infty,0]}(y)\right] = 0 = -\delta_{(-\infty,0]}(\hat{y}) \text{ for all } \hat{y} \in (-\infty, 0].$$

For all $x > 0$, we have

$$xy - \delta_{(-\infty,0]}(y) = \begin{cases} -\infty & \text{if } y > 0, \\ xy & \text{if } y \leq 0. \end{cases}$$

Then $\hat{y} = 0$ is the unique value in $\mathbb{R}$ that maximise $xy - \delta_{(-\infty,0]}(y)$ over all $y \in \mathbb{R}$, and hence

$$0 = x\hat{y} - \delta_{(-\infty,0]}(\hat{y}) = \sup_{y\in\mathbb{R}} \left[xy - \delta_{(-\infty,0]}(y)\right] = v^*(x).$$

Therefore, the conclusion is that

$$v^*(x) = 0 \text{ for all } x \geq 0.$$

3.2 Regret minimisation

Consider an investor with zero initial wealth who faces the liability of meeting a payment $u_t = (u_t^b, u_t^s) \in \mathcal{L}_t^2$ at each time step $t = 0, \ldots, T$. Moreover, he maintains a trading strategy $y = (y_t)_{t=-1}^T$ in the underlying assets, and he liquidates this trading strategy at time T. The investor manages his assets and liabilities by injecting $\phi_t(\Delta y_t + u_t)$ in cash at each time $t = 0, \ldots, T$. His trading strategy is not required to be self-financing and the collection of available trading strategies for the investor is Ψ; see (2.7) for the definition of Ψ.

Remark 3.8. The assumption of zero initial wealth is made without loss of generality. Indeed, if the investor has an initial wealth $w \in \mathbb{R}$ in cash, then the situation is equivalent to that the investor with zero initial wealth facing the liability of meeting a payment $u_0 - (w, 0)$ at time 0 and meeting a payment u_t at each time step $t = 1, \ldots, T$.

For any $t = 0, \ldots, T$, let v_t be a random function such that $v_t^\omega \in \mathbb{V}$ for all $\omega \in \Omega$ and that the function $\omega \mapsto v_t^\omega$ is constant on each node in Ω_t, in other words,

$$v_t^\omega = v_t^{\omega'} \text{ for all } \omega, \omega' \in \nu \text{ and } \nu \in \Omega_t.$$

Observe that v_t is $\mathcal{F}_t$-measurable; see Definition A.16 for the notion of a measurable random function. The investor measures his regret by the quantity $v_t(\phi_t(\Delta y_t + u_t))$ at time step t. His objective is to minimise his expected total regret, in other words, solve the following optimisation problem:

$$\text{minimise } \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta y_t + u_t\right)\right)\right] \text{ over } y \in \Psi. \tag{3.8}$$

The value function $V : \mathcal{N}^2 \to \mathbb{R} \cup \{\infty\}$ of the optimisation problem (3.8) is defined as

$$V\left(u\right) = \inf_{y \in \Psi} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta y_t + u_t\right)\right)\right]. \tag{3.9}$$

We call $\hat{y}$ a solution to (3.8) if $\hat{y} \in \Psi$ and

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta \hat{y}_t + u_t\right)\right)\right] = V(u).$$

Pennanen (2014) considered a similar regret minimisation problem in the market with convex transaction costs. He established the existence of solution for his problem. However, he did not provide computational techniques for solving this problem. In his work, the investor faces the liability of meeting a sequence of cash flows rather than a sequence of portfolios. By letting $u_t = (c_t, 0)$ in (3.8) for some $c_t \in \mathcal{L}_t$ for each $t = 0, \ldots, T$, the problem (3.8) can be written as

$$\text{minimise} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta y_t\right) + c_t\right)\right] \text{ over } y \in \Psi,$$

which is a special example of the regret minimisation problem in Pennanen (2014).

Remark 3.9. By Definition 3.2, regret functions are bounded from below, there exists $a \in \mathbb{R}$ such that

$$v_t^{\omega}(x) \geq a \text{ for all } x \in \mathbb{R},\ t = 0, \ldots, T \text{ and } \omega \in \Omega.$$

This implies that

$$\sum_{t=0}^{T} \mathbb{E}[v_t(x_t)] \geq \sum_{t=0}^{T} \mathbb{E}[a] = a(T+1) \text{ for all } (x_t)_{t=0}^{T} \in \mathcal{N}. \tag{3.10}$$

Therefore, we have $V(u) \geq a(T+1) > -\infty$.

Remark 3.10. If $V(u) = \infty$, then

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta y_t + u_t\right)\right)\right] = \infty \text{ for all } y \in \Psi$$

and hence every element from Ψ is a solution to (3.8). In Corollary 3.16, we will show that there exists a solution to (3.8) in the case when $V(u) < \infty$.

The following example shows that it is possible that $V(u) = \infty$ for some $u \in \mathcal{N}^2$.

Example 3.11. Suppose $v_t = \delta_{(-\infty,0]}$ for all $t = 0, \ldots, T$. By (3.9), we have for any $u = (u_t)_{t=0}^{T} \in \mathcal{N}^2$ that

$$\begin{aligned} V\left(u\right) &= \inf_{y \in \Psi} \sum_{t=0}^{T} \mathbb{E}\left[\delta_{(-\infty,0]}\left(\phi_t\left(\Delta y_t + u_t\right)\right)\right] \\ &= \begin{cases} 0 & \text{if } \exists y \in \Psi :\ \phi_t\left(\Delta y_t + u_t\right) \leq 0\, \forall t = 0, \ldots, T, \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Clearly, when $u_t = 0$ for each $t = 0, \ldots, T$, we have $V(u) = 0$. Now, we take $u_t = 1$ for all $t = 0, \ldots, T$. Suppose by contradiction that $V(u) = 0$. Then there exists $y \in \Psi$ such that $\phi_t\left(\Delta y_t + u_t\right) \leq 0$ for all $t = 0, \ldots, T$. Define

$y' \in \mathcal{N}^{2\prime}$ by

$$\begin{aligned} y'_{-1} &= y_{-1} = 0, \\ y'_t &= y_t + \sum_{k=0}^{t} u_k \text{ for all } t = 0, \dots, T. \end{aligned}$$

It follows that

$$\phi_t \left(\Delta y'_t\right) = \phi_t \left(\Delta y_t + u_t\right) \leq 0 \text{ for all } t = 0, \dots, T,$$

and hence $y' \in \Phi$ and $y'_T \in \mathcal{A}_T$. However, since $y_T = 0$, we have $y'_T = T + 1$. So the no-arbitrage condition is violated and hence $V(u) = \infty$.

The example below gives a connection between the problem (3.8) and optimal investment and consumption problems considered in the financial market models.

Example 3.12. For each $t = 1, \dots, T$, we set $v_t = v_0$, and hence the function $\omega \mapsto v_t^\omega$ is constant on Ω. Moreover, the functions $v_0, \dots, v_T$ are the same. By letting $U(x) = -v_0(-x)$ for all $x \in \mathbb{R}$, we have

$$v_t(x) = -U(-x) \text{ for all } t = 0, \dots, T \text{ and } x \in \mathbb{R}.$$

The process $(u_t)_{t=0}^T$ is defined as $u_0 = (-w, 0)$ for some $w \in \mathbb{R}$, and $u_t = 0$ for all $t = 1, \dots, T$. This means that the investor will receive w amount of cash at time 0, and he has no future liabilities. Then (3.8) can be written as

$$\begin{aligned} &\inf_{y\in\Psi} \textstyle\sum_{t=0}^T \mathbb{E}\left[v_t(\phi_t(\Delta y_t + u_t))\right] \\ &\quad = \inf_{y\in\Psi} \mathbb{E}\left[-\textstyle\sum_{t=0}^T U(-\phi_t(\Delta y_t + u_t))\right] \\ &\quad = -\sup_{y\in\Psi} \mathbb{E}\left[\textstyle\sum_{t=0}^T U(-\phi_t(\Delta y_t + u_t))\right] \\ &\quad = -\sup_{y\in\Psi} \mathbb{E}\left[U(-\phi_0(\Delta y_0 - (w,0))) + \textstyle\sum_{t=1}^T U(-\phi_t(\Delta y_t))\right], \end{aligned} \tag{3.11}$$

where $-\phi_0(\Delta y_0 - (w,0)) = -\phi_0(y_0 - (w,0))$ represents the consumption in cash at time 0 and $-\phi_t(\Delta y_t)$ represents the consumption in cash at time t for all $t = 1, \dots, T$. The maximisation problem in (3.11) is the so-called optimal optimal investment and consumption problem studied e.g. in Abrams & Karmarkar (1980), Cai (2009), Cai, Judd & Xu (2013). In the continuous-time version, there are many papers about optimal investment and consumption with transaction costs; see Davis & Norman (1990), Shreve & Soner (1994), Øksendal & Sulem (2002), Liu (2004), Janeček & Shreve (2004), Hobson &

Zhu (2016).

There is a link between the problem (3.8) and utility maximisation problems considered in the financial market models.

Example 3.13. Let $v_t = \delta_{(-\infty,0]}$ for all $t = 0, \ldots, T-1$. This means that the investor is not allowed to inject any positive amount of cash before the terminal time step T. Then we write $U(x) := -v_T(-x)$ for all $x \in \mathcal{L}_T$, where the function $\omega \mapsto v_T^\omega$ is assumed to be constant on Ω. Moreover, we define $(u_t)_{t=0}^T$ as $u_0 = (-w, 0)$ for some $w \in \mathbb{R}$, $u_T = (c, 0)$ for some $c \in \mathcal{L}_T$, and $u_t = 0$ for all $t = 1, \ldots, T-1$. This implies that the investor has an initial wealth $w \in \mathbb{R}$ in cash at time 0, and that his liability is to deliver $c \in \mathcal{L}_T$ in cash at time T. Then (3.8) can be written as

$$
\begin{aligned}
&\inf_{y \in \Psi} \sum_{t=0}^{T} \mathbb{E}\left[v_t(\phi_t(\Delta y_t + u_t))\right] \\
&= \inf_{y \in \Psi} \left\{ \mathbb{E}\left[v_T(\phi_T(\Delta y_T) + c)\right] \middle| \, \phi_0(\Delta y_0) \leq w, \phi_t(\Delta y_t) \leq 0 \, \forall 0 < t < T \right\} \\
&= \inf_{y \in \Psi} \left\{ -\mathbb{E}\left[U(-\phi_T(\Delta y_T) - c)\right] \middle| \, \phi_0(\Delta y_0) \leq w, \phi_t(\Delta y_t) \leq 0 \, \forall 0 < t < T \right\} \\
&= -\sup_{y \in \Psi} \left\{ \mathbb{E}\left[U(-\phi_T(\Delta y_T) - c)\right] \middle| \, \phi_0(\Delta y_0) \leq w, \phi_t(\Delta y_t) \leq 0 \, \forall 0 < t < T \right\}
\end{aligned}
$$

where $-\phi_T(\Delta y_T) - c = -\phi_T(-y_{T-1}) - c$ is the investor's terminal wealth after liquidation, and $\mathbb{E}[U(-\phi_T(\Delta y_T) - c)]$ represents the investor's expected utility of the terminal wealth. We can set $c = 0$ if the investor has no liabilities at the terminal time T. The maximisation problem above is the so-called utility maximisation problem studied e.g. in Gennotte & Jung (1994), Boyle & Lin (1997), Sass (2005), Cetin & Rogers (2007), Brown & Smith (2011), Atkinson & Quek (2012). In particular, Cetin & Rogers (2007) considered convex transaction costs, and Sass (2005) considered piecewise proportional, fixed and constant costs. There are also many papers studied the utility maximisation problem with transaction costs in continuous-time settings; see Davis, Panas & Zariphopoulou (1993), Cvitanić & Karatzas (1996), Deelstra, Pham & Touzi (2001), Dai & Yi (2009), Bichuch (2012), Czichowsky, Peyre, Schachermayer & Yang (2018).

3.2.1 Existence of solution

In this section, we are going to prove that there exists a solution to the problem (3.8). To achieve this, we first have to rewrite our problem as an unconstrained optimisation problem.

Define $f : \Omega \times \mathbb{R}^{T+1} \times \mathbb{R}^{2(T+2)} \times \mathbb{R}^{2(T+1)} \to \mathbb{R} \cup \{\infty\}$ as

$$f^{\omega}(x, y, u) := \sum_{t=0}^{T} v_t^{\omega}(x_t) + \delta_{B^{\omega}}(x, y, u) \tag{3.12}$$

where $x = (x_0, \dots, x_T)$, $y = (y_{-1}, \dots, y_T)$, $u = (u_0, \dots, u_T)$ and

$$\begin{aligned} B^{\omega} := \Big\{ (x, y, u) \in \mathbb{R}^{T+1} \times \mathbb{R}^{2(T+2)} \times \mathbb{R}^{2(T+1)} \Big| \, & y_{-1} = y_T = 0, \\ & \Delta y_t - (x_t, 0) + u_t \in -\mathcal{K}_t^{\omega} \text{ for all } t = 0, \dots, T \Big\}. \end{aligned}$$

Note that B is a set-valued function, and both f and δ_B are random functions; see Appendix A.3 for definitions of a set-valued function and a random function. Moreover, the set-valued function B is $\mathcal{F}_T$-measurable since $\mathcal{K}_t$ is $\mathcal{F}_t$-measurable for all t; see Definition A.12. We have for all $\omega \in \Omega$ that

$$\begin{aligned} \operatorname{epi} \delta_{B^{\omega}} &= \{(x, y) \mid \delta_{B^{\omega}}(x) \le y, \, y \in \mathbb{R}\} \\ &= \{(x, y) \mid x \in B^{\omega}, \, y \ge 0\} \\ &= B^{\omega} \times [0, \infty). \end{aligned} \tag{3.13}$$

Thus the set-valued function $\omega \mapsto \operatorname{epi} \delta_{B^{\omega}}$ is $\mathcal{F}_T$-measurable and hence the random function δ_B is $\mathcal{F}_T$-measurable; see Definition A.16.

Proposition 3.14. *For any $\omega \in \Omega$, the set B^{ω} is a closed convex cone containing* 0*.*

Proof. Fix any $(x, y, u), (x', y', u') \in B^{\omega}$ and $a, b \ge 0$. Then

$$a y_{-1} + b y'_{-1} = a y_T + b y'_T = 0$$

and

$$\begin{aligned} &\Delta(a y_t + b y'_t) - (a x_t + b x'_t, 0) + a u_t + b u'_t \\ &\qquad = a \Delta y_t - a(x_t, 0) + a u_t + b \Delta y'_t - b(x'_t, 0) + b u'_t \in -\mathcal{K}_t^{\omega} \end{aligned}$$

for all $t = 0, \dots, T$ since $-\mathcal{K}_t^{\omega}$ is a convex cone containing 0. This means that

$$a(x, y, u) + b(x', y', u') \in B^{\omega}$$

and hence $B^{\omega}$is a convex cone that contains 0. It remains to show that B^{ω} is closed. Suppose $(x^{(k)}, y^{(k)}, u^{(k)})_{k \in \mathbb{N}}$ is a sequence in B^{ω} that converges to

$$(x, y, u) \in \mathbb{R}^{T+1} \times \mathbb{R}^{2(T+2)} \times \mathbb{R}^{2(T+1)}.$$

Then we have $x_t^{(k)} \to x_t$, $u_t^{(k)} \to u_t$ for all $t = 0, \ldots, T$ and $y_t^{(k)} \to y_t$ for all $t = -1, \ldots, T$. This implies $y_{-1} = y_T = 0$. For any $t = 0, \ldots, T$, we have $\Delta y_t^{(k)} - (x_t^{(k)}, 0) + u_t^{(k)} \in -\mathcal{K}_t^\omega$ for all $k \in \mathbb{N}$. Since $\mathcal{K}_t^\omega$ is closed and $\Delta y_t^{(k)} - (x_t^{(k)}, 0) + u_t^{(k)} \in -\mathcal{K}_t^\omega$ for all $k \in \mathbb{N}$, we have $\Delta y_t - (x_t, 0) + u_t \in -\mathcal{K}_t^\omega$. Thus $(x, y, u) \in B^\omega$ and hence B^ω is closed. □

Fix any $\omega \in \Omega$. From (3.13) and the fact that B^ω is a closed convex cone, the set $\operatorname{epi} \delta_{B^\omega}$ is also a closed convex cone. Combining this with $0 \in \operatorname{epi} \delta_{B^\omega}$ by (3.13), we have $(\operatorname{epi} \delta_{B^\omega})^\infty = \operatorname{epi} \delta_{B^\omega}$ (Lemma A.3), and hence

$$\delta_{B^\omega}^\infty = \delta_{B^\omega};$$

see Section A.1 for the definition of recession cone and recession function. Notice that δ_{B^ω} is a convex function because its epigraph $\operatorname{epi} \delta_{B^\omega}$ is convex. Then $(x, y, u) \mapsto f^\omega(x, y, u)$ is convex and bounded from below because it is the sum of convex functions that are bounded from below.

As regret functions are nondecreasing, it follows from (3.9) that

$$V(u) = \inf \left\{ \sum_{t=0}^{T} \mathbb{E}\left[v_t(x_t)\right] \middle| x \in \mathcal{N}, y \in \Psi, \phi_t(\Delta y_t + u_t) \le x_t, \; \forall t = 0, \ldots, T \right\}. \tag{3.14}$$

Observe that $\phi_t(\Delta y_t + u_t) \le x_t$ is equivalent to $\phi_t(\Delta y_t - (x_t, 0) + u_t) \le 0$ and hence equivalent to $\Delta y_t - (x_t, 0) + u_t \in -\mathcal{K}_t$ by (2.4). Then (3.12) gives

$$V(u) = \inf \left\{ \mathbb{E}\left[f(x, y, u)\right] \,\middle|\, (x, y) \in \mathcal{N} \times \mathcal{N}^{2\prime} \right\} \text{ for all } u \in \mathcal{N}^2. \tag{3.15}$$

This completes the formulation of the unconstrained optimisation problem.

Since Ω is finite, expectation is a convex combination, and hence the mapping $(x, y, u) \mapsto \mathbb{E}[f(x, y, u)]$ is convex. Thus the function V is also convex (Rockafellar 1974, Theorem 1). Since $f(0, 0, 0) = 0$, it follows that

$$\begin{aligned} V(0) &= \inf \left\{ \mathbb{E}\left[f(x, y, 0)\right] \,\middle|\, (x, y) \in \mathcal{N} \times \mathcal{N}^{2\prime} \right\} \\ &\le f(0, 0, 0) = 0. \end{aligned}$$

Moreover, we have $V(u) > -\infty$ for all $u \in \mathcal{N}^2$ because f is bounded from below. Therefore the function V is proper.

The following result shows that there always exists a solution to the problem (3.15). We refer to Section A.1 for the notion of lower semicontinuity.

Theorem 3.15. *Under the assumption that the robust no-arbitrage condition holds true, the function V is lower semicontinuous on $\mathcal{N}^2$ and the infimum in*

(3.15) *is attained for every* $u \in \mathcal{N}^2$ *such that* $V(u) < \infty$.

Proof. The function f is a convex normal integrand (Rockafellar & Wets 2009, Definition 14.27, Proposition 14.44(c)). The desired result follows from Pennanen & Perkkiö (2012, Theorem 2), provided that the set

$$L := \left\{ (x, y) \in \mathcal{N} \times \mathcal{N}^{2\prime} \middle| f^{\omega\infty}(x(\omega), y(\omega), 0) \le 0 \text{ for all } \omega \in \Omega \right\}$$

is a linear space; see Roman (2008, pp. 35-36) for the definition of a linear space. Thus, it suffices to show that L is a linear space.

Fix any $(x, y, u) \in \mathcal{N} \times \mathcal{N}^{2\prime} \times \mathcal{N}^2$ and $\omega \in \Omega$. For convenience, we shall suppress ω in the remainder of the proof. We have

$$f^{\infty}(x, y, u) = \sum_{t=0}^{T} v_t^{\infty}(x_t) + \delta_B^{\infty}(x, y, u)$$

(Rockafellar 1997, Theorem 9.3), in other words,

$$\begin{aligned} f^{\infty}(x, y, u) &= \sum_{t=0}^{T} \delta_{(-\infty,0]}(x_t) + \delta_B(x, y, u) \\ &= \begin{cases} 0 & \text{if } (x, y, u) \in B,\ x_t \le 0 \text{ for all } t = 0, \dots, T \\ \infty & \text{otherwise.} \end{cases} \end{aligned}$$

Thus

$$\begin{aligned} L &= \left\{ (x, y) \in \mathcal{N} \times \mathcal{N}^{2\prime} \mid (x, y, 0) \in B,\ x_t \le 0 \text{ for all } t = 0, \dots, T \right\} \\ &= \{ (x, y) \in \mathcal{N} \times \Psi \mid x_t \le 0, \Delta y_t - (x_t, 0) \in -\mathcal{K}_t \text{ for all } t = 0, \dots, T \}. \end{aligned}$$

The final step is to show that

$$L = \{ (0, y) \in \mathcal{N} \times \Psi \mid \Delta y_t \in -\mathcal{K}_t \text{ for all } t = 0, \dots, T \} = 0 \times (\Psi \cap \Phi), \quad (3.16)$$

from which it follows that L is a linear space (Proposition 2.8). To this end, fix any $(x, y) \in L$. Suppose by contradiction that $\mathbb{P}(x_{t^*} < 0) > 0$ for some t^*. Define $z = (z_t)_{t=-1}^T \in \mathcal{N}^{2\prime}$ by

$$\begin{aligned} z_{-1} &:= 0, \\ z_t &:= y_t - \sum_{s=0}^{t} (x_s, 0) \text{ for all } t = 0, \dots, T. \end{aligned}$$

Then

$$\Delta z_t = \Delta y_t - (x_t, 0) \in -\mathcal{K}_t \text{ for all } t = 0, \dots, T.$$

This means $z \in \Phi$ and hence $z_T \in \mathcal{A}_T$. However, it follows from $y_T = 0$ that

$$z_T = \sum_{t=0}^{T} (x_t, 0),$$

and hence $z_T \in \mathcal{L}^2_{T+}$ and $z_T \neq 0$. This violates the no-arbitrage condition and hence

$$x_t = 0 \text{ for all } t = 0, \ldots, T.$$

Then (3.16) follows. This completes the proof. □

The lower semicontinuity of V will be used to study the dual problem of (3.8) in Section 3.3. The following corollary implies that Theorem 3.15 can be used to show that there exists a solution to the problem (3.8).

Corollary 3.16. *If $V(u) < \infty$ for some $u \in \mathcal{N}^2$, then there exists $\hat{x} \in \mathcal{N}$ and $\hat{y} \in \mathcal{N}^{2\prime}$ such that*

$$V(u) = \mathbb{E}\left[f\left(\hat{x}, \hat{y}, u\right)\right]. \tag{3.17}$$

Moreover, the trading strategy $\hat{y}$ is a solution to the problem (3.8).

Proof. The first claim follows directly from Theorem 3.15. It is sufficient to show that $\hat{y}$ is a solution to (3.8). Since $V(u) = \mathbb{E}\left[f\left(\hat{x}, \hat{y}, u\right)\right]$ is finite, by (3.12) and the comments preceding (3.15) we have $\hat{y} \in \Psi$ and

$$\phi_t(\Delta \hat{y}_t + u_t) \leq \hat{x}_t \text{ for all } t = 0, \ldots, T. \tag{3.18}$$

From (3.9) and $\hat{y} \in \Psi$, we have

$$V\left(u\right) \leq \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta \hat{y}_t + u_t\right)\right)\right].$$

The opposite inequality also holds. Indeed, combining (3.17), (3.12) and (3.18) together with the fact that regret functions are nondecreasing, it follows that

$$V\left(u\right) = \mathbb{E}\left[f\left(\hat{x}, \hat{y}, u\right)\right] = \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\hat{x}_t\right)\right] \geq \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta \hat{y}_t + u_t\right)\right)\right].$$

The result follows. □

3.2.2 Alternative formulation

It is possible to rewrite the problem (3.8) directly in terms of cash injection at each time step. This will reduce the dimensionality of the control variable, from a two dimensional process to a one dimensional process, and aid in the study of the dual problem. Let $u = (u_t)_{t=0}^T \in \mathcal{N}^2$ for the remainder of this section. Consider the following optimisation problem:

$$\text{minimise} \sum_{t=0}^{T} \mathbb{E}\left[v_t(x_t)\right] \text{ over } x \in A_u \tag{3.19}$$

where

$$A_u = \left\{ (x_t)_{t=0}^T \in \mathcal{N} \middle| \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^T (u_t - (x_t, 0)) \right] \leq 0 \,\forall\, (\mathbb{Q}, S) \in \bar{\mathcal{P}} \right\}. \tag{3.20}$$

Note from Corollary 2.17 that A_u is the collection of cash flows $(x_t)_{t=0}^T \in \mathcal{N}$ such that there exists a superhedging strategy with zero initial endowment for the seller of the flow option $(u_t - (x_t, 0))_{t=0}^T$. We call $\hat{x}$ a solution to (3.19) if $\hat{x} \in A_u$ and

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\hat{x}_t\right)\right] = \inf_{x \in A_u} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(x_t\right)\right].$$

The relationship between the problems (3.8) and (3.19) can be summarised in the following result. This result shows that the optimal values of the problems (3.8) and (3.19) are the same. Moreover, a solution to (3.8) (resp. (3.19)) can be constructed from a solution to (3.19) (resp. (3.8)).

Proposition 3.17. *We have*

$$V(u) = \inf_{x \in A_u} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(x_t\right)\right]. \tag{3.21}$$

If $(\hat{x}_t)_{t=0}^T \in A_u$ is a solution to (3.19)*, then there exists a superhedging strategy $(\hat{z}_t)_{t=-1}^T \in \mathcal{N}^{2\prime}$ with zero initial endowment for the seller of the European option $\sum_{t=0}^T (u_t - (\hat{x}_t, 0))$ and moreover the trading strategy $(\hat{y}_t)_{t=-1}^T \in \mathcal{N}^{2\prime}$ defined by*

$$\hat{y}_{-1} := \hat{z}_{-1} = 0 \text{ and } \hat{y}_t := \hat{z}_t - \sum_{k=0}^{t} (u_k - (\hat{x}_k, 0)) \text{ for all } t = 0, \ldots, T,$$

is a solution to (3.8)*. Conversely, if $(\hat{y}_t)_{t=-1}^T \in \mathcal{N}^{2\prime}$ is a solution to* (3.8)*, then $(\hat{x}_t)_{t=0}^T$ defined by*

$$\hat{x}_t := \phi_t\left(\Delta \hat{y}_t + u_t\right) \text{ for all } t = 0, \ldots, T$$

is a solution to the problem (3.19)*.*

Proof. We first establish (3.21). Fix any $(x_t)_{t=0}^T \in A_u$. There exists a superhedging strategy $z = (z_t)_{t=-1}^T$ with zero initial endowment for the seller of the

European option $\sum_{t=0}^{T}(u_t - (x_t, 0))$; see Corollary 2.15. Define $(y_t)_{t=-1}^{T} \in \mathcal{N}^{2\prime}$ as

$$y_{-1} := z_{-1} = 0 \text{ and } y_t := z_t - \sum_{k=0}^{t}(u_k - (x_k, 0)) \text{ for all } t = 0, \dots, T.$$

We have $z_T = \sum_{t=0}^{T}(u_t - (x_t, 0))$ which implies that $y_T = 0$. Thus $(y_t)_{t=-1}^{T} \in \Psi$; see (2.7). Moreover, for any $t = 0, \dots, T$, we have $\Delta y_t = \Delta z_t - u_t + (x_t, 0)$ and hence

$$\phi_t(\Delta y_t + u_t) = \phi_t(\Delta z_t + (x_t, 0)) = \phi_t(\Delta z_t) + x_t \leq x_t$$

by $z \in \Phi$. Since regret functions are nondecreasing, we have

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t\left(x_t\right)\right] \geq \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta y_t + u_t\right)\right)\right]. \tag{3.22}$$

This means that

$$\inf_{x \in A_u} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(x_t\right)\right] \geq \inf_{y \in \Psi} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta y_t + u_t\right)\right)\right]. \tag{3.23}$$

The opposite inequality also holds true. To this end, fix any $(y_t)_{t=-1}^{T} \in \Psi$. Define $x \in \mathcal{N}$ as

$$x_t := \phi_t\left(\Delta y_t + u_t\right) \text{ for all } t = 0, \dots, T.$$

Note that

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t\left(x_t\right)\right] = \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta y_t + u_t\right)\right)\right]. \tag{3.24}$$

For each $t = 0, \dots, T$, we have $\phi_t(\Delta y_t + u_t - (x_t, 0)) = 0$ which implies that $\Delta y_t + u_t - (x_t, 0) \in -\mathcal{K}_t$ by (2.4). Define $z = (z_t)_{t=-1}^{T} \in \mathcal{N}^{2\prime}$ as

$$z_{-1} := y_{-1} = 0 \text{ and } z_t := y_t + \sum_{k=0}^{t}(u_k - (x_k, 0)) \text{ for all } t = 0, \dots, T.$$

Then

$$\Delta z_t = \Delta y_t + u_t - (x_t, 0) \in -\mathcal{K}_t \text{ for all } t = 0, \dots, T$$

and

$$z_T = y_T + \sum_{t=0}^{T}(u_t - (x_t, 0)) = \sum_{t=0}^{T}(u_t - (x_t, 0))$$

because $y_T = 0$. Thus z is a superhedging strategy with zero initial endowment for the seller of the European option $\sum_{t=0}^{T}(u_t - (x_t, 0))$, and this implies that $x \in A_u$; see Corollary 2.15. Combining this with (3.24), we have

$$\inf_{x \in A_u} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(x_t\right)\right] \leq \inf_{y \in \Psi} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta y_t + u_t\right)\right)\right]. \tag{3.25}$$

Therefore, the equality (3.21) follows from (3.23), (3.25) and (3.9).

Suppose that $(\hat{x}_t)_{t=0}^{T} \in A_u$ is a solution to (3.19). Similar to the first part of the proof, there exists a superhedging strategy $(\hat{z}_t)_{t=-1}^{T}$ for the seller of the European option $\sum_{t=0}^{T}(u_t - (\hat{x}_t, 0))$. Moreover, the trading strategy $(\hat{y}_t)_{t=-1}^{T} \in \mathcal{N}^{2'}$ defined by

$$\hat{y}_{-1} := \hat{z}_{-1} = 0 \text{ and } \hat{y}_t := \hat{z}_t - \sum_{k=0}^{t}(u_k - (\hat{x}_k, 0)) \text{ for all } t = 0, \ldots, T,$$

satisfies $(\hat{y}_t)_{t=-1}^{T} \in \Psi$. Combining (3.21) together with the assumption that $(\hat{x}_t)_{t=0}^{T}$ is a solution to (3.19) and (3.22), it follows that

$$V\left(u\right) = \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\hat{x}_t\right)\right] \geq \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta \hat{y}_t + u_t\right)\right)\right].$$

However, we have from (3.9) that

$$V\left(u\right) \leq \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta \hat{y}_t + u_t\right)\right)\right],$$

and hence

$$V\left(u\right) = \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta \hat{y}_t + u_t\right)\right)\right].$$

Thus the trading strategy $(\hat{y}_t)_{t=-1}^{T}$ is a solution to (3.8). Conversely, suppose that $(\hat{y}_t)_{t=-1}^{T} \in \Psi$ is a solution to (3.8), in other words,

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta \hat{y}_t + u_t\right)\right)\right] = V(u).$$

Define $(\hat{x}_t)_{t=0}^{T} \in \mathcal{N}$ by $\hat{x}_t := \phi_t\left(\Delta \hat{y}_t + u_t\right)$ for all $t = 0, \ldots, T$. Then we have

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\hat{x}_t\right)\right] = \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(\phi_t\left(\Delta \hat{y}_t + u_t\right)\right)\right] = V(u) = \inf_{x \in A_u} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(x_t\right)\right]$$

by (3.21). Similar to the first part of the proof, we have $(\hat{x}_t)_{t=0}^{T} \in A_u$ and hence $(\hat{x}_t)_{t=0}^{T}$ is a solution to the problem (3.19). □

From Remark 3.10 and Corollary 3.16, there always exists a solution to the problem (3.8). Therefore, the result below follows from Proposition 3.17.

Corollary 3.18. *There exists a solution to the problem* (3.19)*.*

The property of V in the lemma below will be used in the proof of Theorem 3.39.

Lemma 3.19. *For any $u' \in \mathcal{N}^2$ such that $0 \in A_{u'}$, we have*

$$V\left(u + u'\right) \leq V\left(u\right).$$

Proof. Fix any $x = (x_t)_{t=0}^T \in A_u$. For all $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$, we have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^T (u_t + u_t' - (x_t, 0))\right] \\ &\qquad = \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^T (u_t - (x_t, 0))\right] + \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^T u_t'\right] \leq 0\end{aligned}$$

because $x \in A_u$ and $0 \in A_{u'}$. This means $x \in A_{u+u'}$. Thus $A_{u+u'} \supseteq A_u$ which implies that

$$\inf_{x \in A_{u+u'}} \sum_{t=0}^T \mathbb{E}\left[v_t(x_t)\right] \leq \inf_{x \in A_u} \sum_{t=0}^T \mathbb{E}\left[v_t(x_t)\right].$$

Then the result follows from (3.21). □

3.3 The dual problem

In this section, we will introduce a Lagrangian dual problem of (3.19). Sometimes the solutions to the dual problem are easier to find and these solutions can be used to construct solutions to the primal problem. First, we define the Lagrangian and present the connection between the Lagrangian function and the problem (3.19). After that, the dual problem of (3.19) is defined by means of the Lagrangian. In Section 3.4, the relationship between the dual problem and the problem (3.19) will be studied in detail.

Fix any $u = (u_t)_{t=0}^T \in \mathcal{N}^2$ for the remainder of this section. We define the Lagrangian $L_u : \mathcal{N} \times [0, \infty) \times \bar{\mathcal{P}} \to \mathbb{R} \cup \{\infty\}$ as

$$L_u(x, \lambda, (\mathbb{Q}, S)) = \sum_{t=0}^T \mathbb{E}[v_t(x_t)] + \lambda \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \sum_{t=0}^T (u_t - (x_t, 0))\right], \tag{3.26}$$

where $x = (x_t)_{t=0}^T \mapsto \sum_{t=0}^T \mathbb{E}[v_t(x_t)]$ is the objective function of the problem (3.19), the value $\mathbb{E}_{\mathbb{Q}}[(1, S_T) \cdot \sum_{t=0}^T (u_t - (x_t, 0))]$ is used in the constraints of (3.19), and λ is a nonnegative number. The formulation of L_u in (3.26) is motivated by (74) of Schachermayer (2002). He studied the Lagrangian dual problem of an utility maximisation problem in an incomplete friction-free market model.

Remark 3.20. Observe from Remark 2.12 that L_u in (3.26) can be written as

$$L_u(x,\lambda,(\mathbb{Q},S)) = \sum_{t=0}^{T} \mathbb{E}\left[v_t(x_t) - \lambda\Lambda_t^{\mathbb{Q}} x_t\right] + \lambda\mathbb{E}_{\mathbb{Q}}\left[(1,S_T)\cdot\sum_{t=0}^{T} u_t\right]; \tag{3.27}$$

see (2.17) for the definition of $(\Lambda_t^{\mathbb{Q}})_{t=0}^T$. This formulation of L_u will be used in the study of the dual problem. Moreover, for any $\lambda \geq 0$ and $(\mathbb{Q},S) \in \bar{\mathcal{P}}$, we have

$$\begin{aligned}
&\inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S)) \\
&\quad= \inf_{x\in\mathcal{N}} \textstyle\sum_{t=0}^T \mathbb{E}\left[v_t(x_t) - \lambda\Lambda_t^{\mathbb{Q}} x_t\right] + \lambda\mathbb{E}_{\mathbb{Q}}\left[(1,S_T)\cdot\sum_{t=0}^T u_t\right] \\
&\quad= -\sup_{x\in\mathcal{N}} \textstyle\sum_{t=0}^T \mathbb{E}\left[\lambda\Lambda_t^{\mathbb{Q}} x_t - v_t(x_t)\right] + \lambda\mathbb{E}_{\mathbb{Q}}\left[(1,S_T)\cdot\sum_{t=0}^T u_t\right] \\
&\quad= -\textstyle\sum_{t=0}^T \sup_{x_t\in\mathcal{L}_t} \mathbb{E}\left[\lambda\Lambda_t^{\mathbb{Q}} x_t - v_t(x_t)\right] + \lambda\mathbb{E}_{\mathbb{Q}}\left[(1,S_T)\cdot\sum_{t=0}^T u_t\right].
\end{aligned} \tag{3.28}$$

As infimum is taken, the value $\inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S))$ in (3.28) only depends on λ and $(\mathbb{Q},S)$. The function $(\lambda,(\mathbb{Q},S)) \mapsto \inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S))$ will be used as the objective function of the dual problem.

Remark 3.21. Fix any $x = (x_t)_{t=0}^T \in \mathcal{N}$, $\lambda \in [0,\infty)$ and $(\mathbb{Q},S) \in \bar{\mathcal{P}}$. Since Ω is finite, we have

$$\left|\lambda\mathbb{E}_{\mathbb{Q}}\left[(1,S_T)\cdot\textstyle\sum_{t=0}^T (u_t - (x_t,0))\right]\right| < \infty.$$

Moreover, regret functions are bounded from below and hence

$$\sum_{t=0}^{T} \mathbb{E}[v_t(x_t)] > -\infty.$$

Then we have from (3.26) that $L_u(x,\lambda,(\mathbb{Q},S)) > -\infty$. However, sometimes the value of L_u can be ∞. For example, in the case when $v_{t^*} = \delta_{(-\infty,0]}$ for some $t^* = 0,\dots,T$, by taking $x_t = 1$ for all $t = 0,\dots,T$, we have $v_{t^*}(x_{t^*}) = \infty$ and hence $L_u(x,\lambda,(\mathbb{Q},S)) = \infty$.

First of all, we consider the following minimisation problem with the objective function $x \mapsto \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S))$ and the feasible set $\mathcal{N}$:

$$\text{minimise} \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S)) \text{ over } x \in \mathcal{N}. \tag{3.29}$$

We say that $\hat{x}$ is a solution to the problem (3.29) if $\hat{x} \in \mathcal{N}$ and

$$\sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) = \inf_{x \in \mathcal{N}} \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)).$$

Remark 3.22. Fix any $x = (x_t)_{t=0}^T \in \mathcal{N}$. We are going to present the value $\sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S))$ by considering the following two cases.

1. Let $x \notin A_u$; see (3.20) for the definition of A_u. Then there exists some $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$ such that

   $$\mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T (u_t - (x_t, 0)) \right] > 0.$$

   The value $\lambda \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \sum_{t=0}^T (u_t - (x_t, 0)) \right]$ can be made arbitrarily large by taking λ arbitrarily large. Thus from (3.26) we have

   $$\sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)) = \infty.$$

2. If $x \in A_u$, then

   $$\mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T (u_t - (x_t, 0)) \right] \leq 0 \text{ for all } (\mathbb{Q}, S) \in \bar{\mathcal{P}}.$$

   Combining this with (3.26), we have

   $$\sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)) = \sum_{t=0}^T \mathbb{E}[v_t(x_t)]. \tag{3.30}$$

The problem (3.29) and the problem (3.19) are equivalent in the following sense.

Proposition 3.23. *The optimal values of the problems* (3.29) *and* (3.19) *coincide, in other words,*

$$\inf_{x \in \mathcal{N}} \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)) = \inf_{x \in A_u} \sum_{t=0}^T \mathbb{E}[v_t(x_t)] = V(u). \tag{3.31}$$

If $\hat{x}$ is a solution to (3.19)*, then it is also a solution to* (3.29)*. Moreover, if $\hat{x}$ is a solution to* (3.29) *and $V(u) < \infty$, then $\hat{x}$ is a solution to* (3.19)*.*

Proof. First, we are going to show that (3.31) holds true. From Remark 3.22.1, we have

$$\inf_{x \in \mathcal{N}} \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)) = \inf_{x \in A_u} \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)). \tag{3.32}$$

Moreover, by taking the infimum over $x \in A_u$ on both sides of (3.30), we have

$$\inf_{x \in A_u} \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)) = \inf_{x \in A_u} \sum_{t=0}^{T} \mathbb{E}[v_t(x_t)]. \tag{3.33}$$

Then (3.31) follows from (3.32), (3.33) and (3.21).

Suppose that $\hat{x}$ is a solution to the problem (3.19). This implies that $\hat{x} \in A_u$. From (3.30) and the fact that $\hat{x}$ is a solution to the problem (3.19), it follows that

$$\sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) = \sum_{t=0}^{T} \mathbb{E}[v_t(\hat{x}_t)] = \inf_{x \in A_u} \sum_{t=0}^{T} \mathbb{E}[v_t(x_t)].$$

Combining this with (3.31), we have

$$\sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) = \inf_{x \in \mathcal{N}} \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)).$$

This means that $\hat{x}$ is a solution to (3.29).

Suppose that $\hat{x}$ is a solution to (3.29) and that $V(u) < \infty$. Since $\hat{x}$ is a solution to (3.29), we have

$$\sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) = \inf_{x \in \mathcal{N}} \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)). \tag{3.34}$$

Combining this with (3.31), we have

$$\sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) = V(u) < \infty,$$

and hence $\hat{x} \in A_u$ by Remark 3.22.1. Moreover, combining (3.30) and (3.34) together with (3.31), it follows that

$$\sum_{t=0}^{T} \mathbb{E}[v_t(\hat{x}_t)] = \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) = \inf_{x \in A_u} \sum_{t=0}^{T} \mathbb{E}\left[v_t\left(x_t\right)\right].$$

Thus $\hat{x}$ is a solution to the problem (3.19). This completes the proof. $\square$

Now, we are going to introduce the dual optimisation problem of (3.19) based on the Lagrangian L_u. For the dual problem, the objective function is $(\lambda, (\mathbb{Q}, S)) \mapsto \inf_{x \in \mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S))$, and the feasible set is $[0, \infty) \times \bar{\mathcal{P}}$. We call the following problem the *dual problem* of (3.19):

$$\text{maximise } \inf_{x \in \mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) \text{ over } (\lambda, (\mathbb{Q}, S)) \in [0, \infty) \times \bar{\mathcal{P}}. \tag{3.35}$$

We call $(\hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S}))$ a solution to the dual problem (3.35) if $\hat{\lambda} \geq 0$, $(\hat{\mathbb{Q}}, \hat{S}) \in \bar{\mathcal{P}}$ and

$$\inf_{x\in\mathcal{N}} L_u(x, \hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S})) = \sup_{\lambda\geq 0, (\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)).$$

For any $t = 0, \ldots, T$, we define the random function v_t^* as

$$v_t^{*\omega}(y) := \sup\{ yx - v_t^\omega(x) | \, x \in \mathbb{R}\} \text{ for all } \omega \in \Omega \text{ and } y \in \mathbb{R}.$$

Observe that $v_t^{*\omega}$ is the *conjugate function* (Rockafellar 1974, (3.10)) of v_t^ω for each $\omega \in \Omega$. Thus the function $\omega \mapsto v_t^{*\omega}$ is constant on each node in Ω_t and hence v_t^* is $\mathcal{F}_t$-measurable; see the comments following Definition A.16. Combining (3.28) with Lemma 3.24 below, the objective function of the dual problem (3.35) can be written as

$$\inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) = -\sum_{t=0}^{T} \mathbb{E}\left[v_t^*(\lambda\Lambda_t^{\mathbb{Q}})\right] + \lambda\mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \sum_{t=0}^{T} u_t\right] \tag{3.36}$$

for all $(\lambda, (\mathbb{Q}, S)) \in [0, \infty) \times \bar{\mathcal{P}}$. The following result will also be used for establishing the connection between problems (3.29) and (3.35) in Theorem 3.31.

Lemma 3.24. *For any $t = 0, \ldots, T$ and $y \in \mathcal{L}_t$, we have*

$$\mathbb{E}\left[v_t^*(y)\right] = \sup\left\{ \mathbb{E}\left[yx - v_t(x)\right] | \, x \in \mathcal{L}_t \right\}.$$

Proof. Fix any $t = 0, \ldots, T$ and $y \in \mathcal{L}_t$. For any $x \in \mathcal{L}_t$, we have

$$\mathbb{E}\left[yx - v_t(x)\right] = \sum_{\nu\in\Omega_t} \mathbb{P}(\nu) \left(y(\nu)x(\nu) - v_t^\nu(x(\nu))\right) \tag{3.37}$$

because x and y are $\mathcal{F}_t$-measurable and the function $\omega \mapsto v_t^\omega$ is constant on each node in Ω_t; see the comments following Definition A.16 for v_t^ν. Taking supremum over $x \in \mathcal{L}_t$ on both sides of (3.37), it yields

$$\sup_{x\in\mathcal{L}_t} \mathbb{E}\left[yx - v_t(x)\right] = \sup_{x\in\mathcal{L}_t} \sum_{\nu\in\Omega_t} \mathbb{P}(\nu) \left(y(\nu)x(\nu) - v_t^\nu(x(\nu))\right) \tag{3.38}$$

The number of nodes in Ω_t is finite, and we denote it by $|\Omega_t|$. The optimisation problem

$$\sup_{x\in\mathcal{L}_t} \sum_{\nu\in\Omega_t} \mathbb{P}(\nu) \left(y(\nu)x(\nu) - v_t^\nu(x(\nu))\right)$$

in (3.38) splits into $|\Omega_t|$ independent optimisation problems over $\mathbb{R}$:

$$\sup_{z\in\mathbb{R}} \left(y(\nu)z - v_t^\nu(z)\right), \text{ where } \nu \in \Omega_t.$$

Thus

$$\sup_{x\in\mathcal{L}_t}\mathbb{E}\left[yx-v_t(x)\right]=\sum_{\nu\in\Omega_t}\mathbb{P}(\nu)\sup_{z\in\mathbb{R}}\left(y(\nu)z-v_t^{\nu}(z)\right)$$
$$=\sum_{\nu\in\Omega_t}\mathbb{P}(\nu)v_t^{*\nu}(y(\nu))=\mathbb{E}\left[v_t^*(y)\right]$$

which completes the proof. □

Remark 3.25. Fix any $(\lambda,(\mathbb{Q},S))\in[0,\infty)\times\bar{\mathcal{P}}$. The value

$$\inf_{x\in\mathcal{N}}L_u(x,\lambda,(\mathbb{Q},S))$$

in (3.36) is finite. Indeed, for any $t=0,\dots,T$, we have for each $\omega\in\Omega$ that $\lambda\Lambda_t^{\mathbb{Q}}(\omega)\geq 0$. Then it follows from (3.4) that $v_t^{*\omega}(\lambda\Lambda_t^{\mathbb{Q}}(\omega))\in[0,\infty)$. Since Ω is finite, both

$$-\textstyle\sum_{t=0}^{T}\mathbb{E}\left[v_t^*(\lambda\Lambda_t^{\mathbb{Q}})\right]\text{ and }\lambda\mathbb{E}_{\mathbb{Q}}\left[(1,S_T)\cdot\textstyle\sum_{t=0}^{T}u_t\right]$$

are finite. Thus $\inf_{x\in\mathcal{N}}L_u(x,\lambda,(\mathbb{Q},S))$ is finite by (3.36).

In Example 3.26 below, we will derive an explicit formula for v_t^* for each $t=0,\dots,T$. Then by using (3.36), we will provide an explicit formula for the objective function

$$(\lambda,(\mathbb{Q},S))\mapsto\inf_{x\in\mathcal{N}}L_u(x,\lambda,(\mathbb{Q},S))$$

of the dual problem.

Example 3.26. From the formulation of the problem (3.8), the value $v_t(x)$ represents the investor's regret after injecting x in cash at each time step $t=0,\dots,T$. Firstly, we are going to specify $(v_t)_{t=0}^T$. Let

$$\mathcal{I}:=\{t_1,\dots,t_n,T\}\subseteq\{0,\dots,T\}$$

be a collection of time steps. Moreover, for all $t\in 0,\dots,T$, $\omega\in\Omega$ and $x\in\mathbb{R}$, we define

$$v_t^{\omega}(x):=\begin{cases}e^{\alpha_t x}-1 & \text{if } t\in\mathcal{I},\\ \delta_{(-\infty,0]}(x) & \text{if } t\notin\mathcal{I}.\end{cases}\tag{3.39}$$

From Examples 3.7.1 and 3.7.3, we have for any $t=0,\dots,T$, $\omega\in\Omega$ and $x\geq 0$ that

$$v_t^{*\omega}(x)=\begin{cases}\frac{x}{\alpha_t}\ln\frac{x}{\alpha_t}-\frac{x}{\alpha_t}+1 & \text{if } t\in\mathcal{I},\\ 0 & \text{if } t\notin\mathcal{I};\end{cases}$$

we always assume that $0\ln 0 = 0$ in this thesis. Combining this with (3.36), for every $(\lambda, (\mathbb{Q}, S)) \in [0, \infty) \times \bar{\mathcal{P}}$, we have

$$\inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) \\ = \lambda \mathbb{E}_{\mathbb{Q}}\left[(1, S_T)\cdot \sum_{t=0}^{T} u_t\right] - \sum_{t\in\mathcal{I}} \mathbb{E}\left[\frac{\lambda\Lambda_t^{\mathbb{Q}}}{\alpha_t}\ln\frac{\lambda\Lambda_t^{\mathbb{Q}}}{\alpha_t} - \frac{\lambda\Lambda_t^{\mathbb{Q}}}{\alpha_t} + 1\right].$$

This gives an explicit presentation of $(\lambda, (\mathbb{Q}, S)) \mapsto \inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S))$.

3.4 The strong duality

In this section, we will study the relationship between the problem (3.19) and and its dual problem (3.35).

Suppose that $(\hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S})) \in [0, \infty) \times \bar{\mathcal{P}}$ is a solution to the problem (3.35). Observe that $\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}} \geq 0$ for all $t = 0, \ldots, T$. We are interested in whether there exists $\hat{x} = (\hat{x}_t)_{t=0}^{T} \in \mathcal{N}$ such that

$$\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}}\hat{x}_t - v_t(\hat{x}_t) = v_t^*(\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}}) \text{ for all } t = 0, \ldots, T. \tag{3.40}$$

If such $\hat{x}$ exists, then it is possible that $\hat{x}$ is a solution to the problem (3.19); see Propositions 3.33 and 3.34 below.

Remark 3.27. In the situation when $\hat{\lambda}\Lambda_{t'}^{\hat{\mathbb{Q}}}(\omega') = 0$ and $v_{t'}^{\omega'}$ is an exponential regret function for some $t' = 0, \ldots, T$ and $\omega' \in \Omega$, there exists no $(\hat{x}_t)_{t=0}^{T} \in \mathcal{N}$ such that (3.40) holds true. This is because that there is no $x \in \mathbb{R}$ such that $0 \times x - v_{t'}^{\omega'}(x) = v_{t'}^{*\omega'}(0)$; see Example 3.7.1.

The following proposition implies that there exists $(\hat{x}_t)_{t=0}^{T} \in \mathcal{N}$ such that (3.40) holds true as long as $(\hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S})) \in (0, \infty) \times \mathcal{P}$ (i.e. $\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}} > 0$ for every $t = 0, \ldots, T$)

Proposition 3.28. *Let $\lambda > 0$ and $(\mathbb{Q}, S) \in \mathcal{P}$. There exists $(x_t)_{t=0}^{T} \in \mathcal{N}$ such that*

$$\lambda\Lambda_t^{\mathbb{Q}} x_t - v_t(x_t) = v_t^*(\lambda\Lambda_t^{\mathbb{Q}}) \textit{ for all } t = 0, \ldots, T. \tag{3.41}$$

Proof. Since $\lambda > 0$ and $(\mathbb{Q}, S) \in \mathcal{P}$, we have $\lambda\Lambda_t^{\mathbb{Q}} > 0$ for all $t = 0, \ldots, T$. We can construct a process $(x_t)_{t=0}^{T} \in \mathcal{N}$ that satisfies (3.41) as follows. Fix any $t = 0, \ldots, T$. Observe that $\Lambda_t^{\mathbb{Q}}$ is $\mathcal{F}_t$-measurable random variable, and the functions $\omega \mapsto v_t^{\omega}$ and $\omega \mapsto v_t^{*\omega}$ are constant on each node in Ω_t. For any $\nu \in \Omega_t$, we have $\lambda\Lambda_t^{\mathbb{Q}}(\nu) > 0$. Moreover, it follows from Proposition 3.6 that there exists $\delta_t(\nu) \in \mathbb{R}$ such that

$$\lambda\Lambda_t^{\mathbb{Q}}(\nu)\delta_t(\nu) - v_t^{\nu}(\delta_t(\nu)) = v_t^{*\nu}(\lambda\Lambda_t^{\mathbb{Q}}(\nu))$$

Then we define

$$x_t(\omega) := \delta_t(\nu) \text{ for all } \omega \in \nu.$$

Notice that, for each $t = 0, \ldots, T$, the value $x_t(\omega)$ is defined for all $\omega \in \Omega$. Moreover, the values of x_t remain unchanged on every node in Ω_t, which means $x_t \in \mathcal{L}_t$. We can conclude that

$$\lambda \Lambda_t^{\mathbb{Q}}(\nu) x_t(\nu) - v_t^{\nu}(x_t(\nu)) = v_t^{*\nu}(\lambda \Lambda_t^{\mathbb{Q}}(\nu)) \text{ for all } t = 0, \ldots, T \text{ and } \nu \in \Omega_t$$

in other words, the condition (3.41) is satisfied. $\square$

Remark 3.29. Let $(\lambda, (\mathbb{Q}, S)) \in [0, \infty) \times \bar{\mathcal{P}}$. It is possible that (3.41) holds true for some $(x_t)_{t=0}^T \in \mathcal{N}$ even if $(\lambda, (\mathbb{Q}, S)) \notin (0, \infty) \times \mathcal{P}$. For example, let $v_t = \delta_{(-\infty, 0]}$ for all $t = 0, \ldots, T$. Then we define $(x_t)_{t=0}^T \in \mathcal{N}$ as $x_t = 0$ for all $t = 0, \ldots, T$. From Example 3.7.3, for all $\omega \in \Omega$ and $t = 0, \ldots, T$, we always have

$$\lambda \Lambda_t^{\mathbb{Q}}(\omega) x_t(\omega) - v_t^{\omega}(x_t(\omega)) = v_t^{*\omega}(\lambda \Lambda_t^{\mathbb{Q}}(\omega)).$$

This means that the process $(x_t)_{t=0}^T$ satisfies (3.41) even if $\lambda \Lambda_{t'}^{\mathbb{Q}}(\omega') = 0$ for some $t' = 0, \ldots, T$ and $\omega' \in \Omega$ (i.e. $(\lambda, (\mathbb{Q}, S)) \notin (0, \infty) \times \mathcal{P}$).

The following auxiliary result will be used in the proofs of Propositions 3.33 and 3.34.

Proposition 3.30. *Fix any $\lambda \geq 0$, $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$ and $\hat{x} = (\hat{x}_t)_{t=0}^T \in \mathcal{N}$. Then we have*

$$\lambda \Lambda_t^{\mathbb{Q}} \hat{x}_t - v_t(\hat{x}_t) = v_t^*(\lambda \Lambda_t^{\mathbb{Q}}) \text{ for all } t = 0, \ldots, T \tag{3.42}$$

if and only if

$$L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) = \inf_{x \in \mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)). \tag{3.43}$$

Proof. Suppose that (3.42) holds true. It follows from (3.27) and (3.42) that

$$\begin{aligned} L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) &= \sum_{t=0}^T \mathbb{E}\left[v_t(\hat{x}_t) - \lambda \Lambda_t^{\mathbb{Q}} \hat{x}_t\right] + \lambda \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \sum_{t=0}^T u_t\right] \\ &= -\sum_{t=0}^T \mathbb{E}\left[v_t^*(\lambda \Lambda_t^{\mathbb{Q}})\right] + \lambda \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \sum_{t=0}^T u_t\right]. \end{aligned}$$

Then (3.36) gives

$$L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) = \inf_{x \in \mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)).$$

Thus (3.43) holds true.

Conversely, suppose that (3.43) holds true. Then (3.43) and (3.27) imply

$$\begin{aligned}\inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S)) &= L_u(\hat{x},\lambda,(\mathbb{Q},S))\\ &= \sum_{t=0}^{T}\mathbb{E}\left[v_t(\hat{x}_t)-\lambda\Lambda_t^{\mathbb{Q}}\hat{x}_t\right]+\lambda\mathbb{E}_{\mathbb{Q}}\left[(1,S_T)\cdot\textstyle\sum_{t=0}^{T}u_t\right].\end{aligned}$$

Combining this with the formulation of $\inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S))$ in (3.36), it follows that

$$\sum_{t=0}^{T}\mathbb{E}\left[v_t(\hat{x}_t)-\lambda\Lambda_t^{\mathbb{Q}}\hat{x}_t+v_t^*(\lambda\Lambda_t^{\mathbb{Q}})\right]=0.$$

For all $t=0,\dots,T$ and $\omega\in\Omega$, it follows from Remark 3.5 that

$$v_t^{*\omega}(\lambda\Lambda_t^{\mathbb{Q}}(\omega))\geq\lambda\Lambda_t^{\mathbb{Q}}(\omega)\hat{x}_t(\omega)-v_t^{\omega}(\hat{x}_t(\omega)),$$

in other words,

$$v_t^{\omega}(\hat{x}_t(\omega))-\lambda\Lambda_t^{\mathbb{Q}}(\omega)\hat{x}_t(\omega)+v_t^{*\omega}(\lambda\Lambda_t^{\mathbb{Q}}(\omega))\geq 0.$$

Thus

$$v_t(\hat{x}_t)-\lambda\Lambda_t^{\mathbb{Q}}\hat{x}_t+v_t^*(\lambda\Lambda_t^{\mathbb{Q}})=0 \text{ for all } t=0,\dots,T,$$

and hence (3.42) follows. □

Regarding the optimisation problems (3.29) and (3.35), the following *weak duality* relation (cf. Bertsekas (2015, p. 3))

$$\inf_{x\in\mathcal{N}}\sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S))\geq\sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}}\inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S))$$

always holds true. The next result shows that this inequality holds true with equality, in other words, the *strong duality* (cf. Bertsekas (2015, p. 3)) also holds true. This result will be used to construct a solution to the problem (3.19) from a solution to problem (3.35).

Theorem 3.31. *Under the assumption that the robust no-arbitrage condition holds true, we have*

$$V(u)=\inf_{x\in\mathcal{N}}\sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S))=\sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}}\inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S)).$$

The proof of Theorem 3.31 above is provided at the end of this section. Moreover, this theorem does not rely on any result in the remainder of this section. Combining (3.21) and Theorem 3.31, the optimal values of the problems (3.19) and (3.35) are the same.

The corollary below follows from Theorem 3.31. Moreover, this corollary will be used to establish Propositions 3.33 and 3.34 below.

Corollary 3.32. *Suppose that $\hat{x}$ is a solution to the problem* (3.19) *and that* $(\hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S}))$ *is a solution to the problem* (3.35). *Then*

$$\sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) = L_u(\hat{x}, \hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S})) = \inf_{x \in \mathcal{N}} L_u(x, \hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S})).$$

Proof. Since $\hat{x}$ is a solution to (3.19), Proposition 3.23 implies that $\hat{x}$ is also a solution to the problem (3.29). Thus

$$\begin{aligned} \inf_{x \in \mathcal{N}} \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S)) &= \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(\hat{x}, \lambda, (\mathbb{Q}, S)) \\ &\geq L_u(\hat{x}, \hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S})) \\ &\geq \inf_{x \in \mathcal{N}} L_u(x, \hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S})) \\ &= \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} \inf_{x \in \mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) \end{aligned}$$

because $(\hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S}))$ is a solution to the problem (3.35). The result follows from Theorem 3.31. □

Propositions 3.33 and 3.34 below use the strong duality to show that it is possible to derive a solution to (3.19) from a solution to (3.35).

Proposition 3.33. *Assume that* $(\hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S}))$ *is a solution to the problem* (3.35), *and let* $\hat{x} = (\hat{x}_t)_{t=0}^T \in \mathcal{N}$. *Then $\hat{x}$ is a solution to the problem* (3.19) *if and only if $\hat{x} \in A_u$ and $\hat{x}$ satisfies*

$$\hat{\lambda} \Lambda_t^{\hat{\mathbb{Q}}} \hat{x}_t - v_t(\hat{x}_t) = v_t^*(\hat{\lambda} \Lambda_t^{\hat{\mathbb{Q}}}) \text{ for all } t = 0, \ldots, T \tag{3.44}$$

and

$$\hat{\lambda} \mathbb{E}_{\hat{\mathbb{Q}}} \left[(1, \hat{S}_T) \cdot \sum_{t=0}^T (u_t - (\hat{x}_t, 0) \right] = 0. \tag{3.45}$$

Proof. Suppose that $\hat{x}$ solves (3.19). Then $\hat{x} \in A_u$. From Corollary 3.32, we have

$$L_u(\hat{x}, \hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S}) = \inf_{x \in \mathcal{N}} L_u(x, \hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S}).$$

Then Proposition 3.30 implies (3.44). Since $\hat{x}$ solves (3.19), we have

$$\sum_{t=0}^T \mathbb{E}[v_t(\hat{x}_t)] = \inf_{x \in A_u} \sum_{t=0}^T \mathbb{E}[v_t(x_t)] = \inf_{x \in \mathcal{N}} \sup_{\lambda \geq 0, (\mathbb{Q}, S) \in \bar{\mathcal{P}}} L_u(x, \lambda, (\mathbb{Q}, S))$$

by (3.31). Moreover, from Proposition 3.23, the process $\hat{x}$ is also a solution to the problem (3.29), in other words,

$$\sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(\hat{x},\lambda,(\mathbb{Q},S)) = \inf_{x\in\mathcal{N}} \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S)).$$

Thus, it follows that

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t(\hat{x}_t)\right] = \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(\hat{x},\lambda,(\mathbb{Q},S)) = L_u(\hat{x},\hat{\lambda},(\hat{\mathbb{Q}},\hat{S})$$

by Corollary 3.32. Combining this with (3.26), the condition (3.45) holds true.

Suppose that $\hat{x}\in A_u$ and (3.44) and (3.45) hold true. Then it follows from (3.26) and Proposition 3.30, that

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t(\hat{x}_t)\right] = L_u(\hat{x},\hat{\lambda},(\hat{\mathbb{Q}},\hat{S})) = \inf_{x\in\mathcal{N}} L_u(x,\hat{\lambda},(\hat{\mathbb{Q}},\hat{S})).$$

Since $(\hat{\lambda},(\hat{\mathbb{Q}},\hat{S}))$ is a solution to the problem (3.35), it follows that

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t(\hat{x}_t)\right] = \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S)).$$

Thus, combining Theorem 3.31 and (3.31), we have

$$\sum_{t=0}^{T} \mathbb{E}\left[v_t(\hat{x}_t)\right] = \inf_{x\in\mathcal{N}} \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} L_u(x,\lambda,(\mathbb{Q},S)) = \inf_{x\in A_u} \sum_{t=0}^{T} \mathbb{E}[v_t(x_t)].$$

This means that $\hat{x}$ is a solution to the problem (3.19). ☐

If a solution to the problem (3.35) can be found and the conditions in the following result are satisfied, then we can use this solution to construct the unique solution to the problem (3.19).

Proposition 3.34. *Suppose that* $(\hat{\lambda},(\hat{\mathbb{Q}},\hat{S}))$ *is a solution to the problem* (3.35) *and that there exists a unique* $\hat{x}=(\hat{x}_t)_{t=0}^{T}\in\mathcal{N}$ *such that*

$$\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}}\hat{x}_t - v_t(\hat{x}_t) = v_t^*(\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}}) \text{ for all } t=0,\dots,T. \tag{3.46}$$

Then $\hat{x}$ *is the unique solution to the problem* (3.19)*.*

Proof. From Corollary 3.18, there exists a solution to the problem (3.19). Observe that for every solution $\bar{x}=(\bar{x}_t)_{t=0}^{T}\in\mathcal{N}$ to the problem (3.19) we have

$$L_u(\bar{x},\hat{\lambda},(\hat{\mathbb{Q}},\hat{S})) = \inf_{x\in\mathcal{N}} L_u(x,\hat{\lambda},(\hat{\mathbb{Q}},\hat{S}))$$

(Corollary 3.32), and this implies that

$$\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}}\bar{x}_t - v_t(\bar{x}_t) = v_t^*(\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}}) \text{ for all } t = 0, \dots, T$$

(Proposition 3.30). However, the solution to (3.46) is unique. This means that we must have $\hat{x} = \bar{x}$ and moreover $\hat{x}$ is the unique solution to the problem (3.19). $\square$

In the case when $(\hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S})) \in (0, \infty) \times \mathcal{P}$ is a solution to (3.35), under the regret functions $(v_t)_{t=0}^T$ defined in Example 3.26, the following example presents the solution to the problem (3.19) in terms of $\hat{\lambda}$ and $\hat{\mathbb{Q}}$ by applying Proposition 3.34.

Example 3.35. Consider the regret function $(v_t)_{t=0}^T$ defined in Example 3.26. Suppose that $(\hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S})) \in (0, \infty) \times \mathcal{P}$ is a solution to the optimisation problem (3.35); we will discuss the existence of such $(\hat{\lambda}, (\hat{\mathbb{Q}}, \hat{S}))$ in Sections 5.1-5.2. Since $\hat{\lambda} > 0$ and $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$, we have $\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}} > 0$ for all $t = 0, \dots, T$. Define $(\hat{x}_t)_{t=0}^T \in \mathcal{N}$ as

$$\hat{x}_t := \begin{cases} \frac{1}{\alpha_t} \ln \frac{\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}}}{\alpha_t} & \text{if } t \in \mathcal{I}, \\ 0 & \text{if } t \in \{0, \dots, T\} \setminus \mathcal{I}, \end{cases}$$

where $\mathcal{I}$ is defined in Example 3.26. Then Examples 3.7.1 and 3.7.3 implies that $(\hat{x}_t)_{t=0}^T$ is the unique process in $\mathcal{N}$ such that

$$\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}}\hat{x}_t - v_t(\hat{x}_t) = v_t^*(\hat{\lambda}\Lambda_t^{\hat{\mathbb{Q}}}) \text{ for all } t = 0, \dots, T.$$

Then Proposition 3.34 implies that $(\hat{x}_t)_{t=0}^T$ is the unique solution to the problem (3.19).

This section ends with the proof of Theorem 3.31 below.

Proof of Theorem 3.31. Firstly, we define the *conjugate function* (Rockafellar 1974, (3.10)) of V as

$$V^*(z) := \sup_{u \in \mathcal{N}^2} \left\{ \sum_{t=0}^T \mathbb{E}\left[z_t \cdot u_t\right] - V(u) \right\} \text{ for all } z = (z_t)_{t=0}^T \in \mathcal{N}^2.$$

From (3.14) and the comments following it, we have for all $z = (z^b, z^s) \in \mathcal{N}^2$ that

$$V^*(z) = \sup_{u \in \mathcal{N}^2} \left\{ \sum_{t=0}^T \mathbb{E}\left[z_t \cdot u_t\right] - \inf \left\{ \sum_{t=0}^T \mathbb{E}\left[v_t(x_t)\right] \middle| (x, y) \in \mathcal{N} \times \Psi, \right.\right.$$

$$\Delta y_t - (x_t, 0) + u_t \in -\mathcal{K}_t \, \forall t = 0, \dots, T \Bigg\}\Bigg\}.$$

After rearrangement, it yields

$$V^*(z) = \sup \Bigg\{ \sum_{t=0}^{T} \mathbb{E}\left[z_t \cdot u_t - v_t(x_t)\right] \Bigg| \, (x, y, u) \in \mathcal{N} \times \Psi \times \mathcal{N}^2,$$

$$\Delta y_t - (x_t, 0) + u_t \in -\mathcal{K}_t \, \forall t = 0, \dots, T \Bigg\}.$$

For all $t = 0, \dots, T$, making change of variable $w_t = \Delta y_t - (x_t, 0) + u_t$, it follows that

$$\begin{aligned} z_t \cdot u_t - v_t(x_t) &= z_t \cdot (w_t - \Delta y_t + (x_t, 0)) - v_t(x_t) \\ &= z_t \cdot w_t - z_t \cdot \Delta y_t + z_t^b x_t - v_t(x_t). \end{aligned}$$

Then

$$V^*(z) = \sup \Bigg\{ \sum_{t=0}^{T} \mathbb{E}\left[z_t \cdot w_t\right] - \sum_{t=0}^{T} \mathbb{E}\left[z_t \cdot \Delta y_t\right] + \sum_{t=0}^{T} \mathbb{E}\left[z_t^b x_t - v_t(x_t)\right] \Bigg|$$

$$(x, y, w) \in \mathcal{N} \times \Psi \in \mathcal{N}^2, w_t \in -\mathcal{K}_t \, \forall t = 0, \dots, T \Bigg\}.$$

This optimisation problem can be decoupled into three optimisation problems over w, y and x, respectively.

Firstly, since $\mathbb{P}(\omega) > 0$ for all $\omega \in \Omega$, we have for all $t = 0, \dots, T$ that

$$\sup_{w_t \in -\mathcal{K}_t} \mathbb{E}\left[z_t \cdot w_t\right] = \begin{cases} 0 & \text{if } z_t \in \mathcal{K}_t^+, \\ \infty & \text{otherwise;} \end{cases}$$

see (2.15) together with (2.14) for the definition of $\mathcal{K}_t^+$; This means

$$\sup \left\{ \sum_{t=0}^{T} \mathbb{E}\left[z_t \cdot w_t\right] \Bigg| \, w \in \mathcal{N}^2, w_t \in -\mathcal{K}_t \, \forall t \right\} = \begin{cases} 0 & \text{if } z_t \in \mathcal{K}_t^+ \, \forall t, \\ \infty & \text{otherwise.} \end{cases} \tag{3.47}$$

Secondly, for all $y = (y_t)_{t=-1}^{T} \in \Psi$, observe from $y_{-1} = y_T = 0$ that

$$-\sum_{t=0}^{T} z_t \cdot \Delta y_t = \sum_{t=1}^{T} z_t \cdot y_{t-1} - \sum_{t=0}^{T-1} z_t \cdot y_t$$

$$
\begin{aligned}
&= \sum_{t=0}^{T-1} z_{t+1} \cdot y_t - \sum_{t=0}^{T-1} z_t \cdot y_t \\
&= \sum_{t=0}^{T-1} \Delta z_{t+1} \cdot y_t.
\end{aligned}
$$

Moreover, for all $t = 0, \ldots, T-1$, the tower property of conditional expectation gives

$$
\begin{aligned}
\sup_{y_t \in \mathcal{L}_t^2} \mathbb{E}\left[\Delta z_{t+1} \cdot y_t\right] &= \sup_{y_t \in \mathcal{L}_t^2} \mathbb{E}\left[\mathbb{E}\left[\Delta z_{t+1} \mid \mathcal{F}_t\right] \cdot y_t\right] \\
&= \begin{cases} 0 & \text{if } \mathbb{E}\left[\Delta z_{t+1} \mid \mathcal{F}_t\right] = 0, \\ \infty & \text{otherwise.} \end{cases}
\end{aligned}
$$

This implies

$$
\sup_{y \in \Psi} \sum_{t=0}^{T} \mathbb{E}\left[-z_t \cdot \Delta y_t\right] = \begin{cases} 0 & \text{if } z \text{ is a martingale,} \\ \infty & \text{otherwise.} \end{cases} \tag{3.48}
$$

Thirdly, notice that

$$
\begin{aligned}
\sup_{x \in \mathcal{N}} \sum_{t=0}^{T} \mathbb{E}\left[z_t^b x_t - v_t(x_t)\right] &= \sum_{t=0}^{T} \sup_{x_t \in \mathcal{L}_t} \mathbb{E}\left[z_t^b x_t - v_t(x_t)\right] \\
&= \sum_{t=0}^{T} \mathbb{E}\left[v_t^*(z_t^b)\right]
\end{aligned} \tag{3.49}
$$

by Lemma 3.24.

Therefore, combining (3.47)-(3.49), it follows that

$$
V^*(z) = \begin{cases} \sum_{t=0}^{T} \mathbb{E}\left[v_t^*(z_t^b)\right] & \text{if } z \in \bar{\mathcal{C}}, \\ \infty & \text{otherwise;} \end{cases} \tag{3.50}
$$

see (2.16) for the definition of $\bar{\mathcal{C}}$.

From Theorem 3.15 and the comments following (3.15), the function V is lower semicontinuous and convex on $\mathcal{N}^2$. Then Theorem 5 of Rockafellar (1974) states that V is equal to its *biconjugate function* (Rockafellar 1974, (3.12)), in other words,

$$
V(u) = \sup_{z \in \mathcal{N}^2} \left\{ \sum_{t=0}^{T} \mathbb{E}\left[u_t \cdot z_t\right] - V^*(z) \right\} \text{ for all } u = (u_t)_{t=0}^{T} \in \mathcal{N}^2.
$$

Fix any $u = (u_t)_{t=0}^T \in \mathcal{N}^2$. It follows from (3.50) and Lemma 2.13 that

$$\begin{aligned} V(u) &= \sup_{z\in\bar{\mathcal{C}}} \sum_{t=0}^T \mathbb{E}\left[u_t \cdot z_t - v_t^*(z_t^b)\right] \\ &= \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} \sum_{t=0}^T \mathbb{E}\left[u_t \cdot \left(\lambda(1,S_t)\Lambda_t^{\mathbb{Q}}\right) - v_t^*(\lambda\Lambda_t^{\mathbb{Q}})\right] \\ &= \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} \left\{ -\sum_{t=0}^T \mathbb{E}\left[v_t^*(\lambda\Lambda_t^{\mathbb{Q}})\right] + \lambda \sum_{t=0}^T \mathbb{E}_{\mathbb{Q}}\left[u_t \cdot (1,S_t)\right]\right\}. \end{aligned}$$

For any $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$, the martingale property of $S = (S_t)_{t=0}^T$ gives

$$\begin{aligned} \sum_{t=0}^T \mathbb{E}_{\mathbb{Q}}\left[u_t \cdot (1,S_t)\right] &= \sum_{t=0}^T \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[u_t \cdot (1,S_T) \middle| \mathcal{F}_t\right]\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[\sum_{t=0}^T u_t \cdot (1,S_T)\right]. \end{aligned}$$

Thus, we have

$$\begin{aligned} V(u) &= \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} \left\{ -\sum_{t=0}^T \mathbb{E}\left[v_t^*(\lambda\Lambda_t^{\mathbb{Q}})\right] + \lambda \mathbb{E}_{\mathbb{Q}}\left[\sum_{t=0}^T u_t \cdot (1,S_T)\right]\right\} \\ &= \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S)) \end{aligned}$$

by (3.36). Combining this with Proposition 3.23, the result follows. □

3.5 Indifference pricing

Consider an investor who is entitled to receive a portfolio $\bar{c}_t \in \mathcal{L}_t^2$ at each time step $t = 0, \ldots, T$. We refer to this sequence of portfolios $\bar{c} = (\bar{c}_t)_{t=0}^T$ as the endowment of the investor. Here negative endowment is interpreted as liability. The process $\bar{c}$ is always considered as a given data. For example, in the situation when the investor's endowment consists of a number of different flow options at time 0, the value $\bar{c}_t$ is the total portfolio that will be received at time t. If the investor is not going to deliver or receive additional portfolios, then $V(-\bar{c})$ is used to represent his minimal regret. We are going to introduce the concepts of seller's and buyer's regret indifference prices of a flow option $c = (c_t)_{t=0}^T \in \mathcal{N}^2$.

Consider the situation when the investor is selling the flow option c. He receives $\delta \in \mathbb{R}$ in cash at time 0, and delivers the portfolio c_t at each time step $t = 0, \ldots, T$. By selling c, the investor's minimal regret becomes $V\left(c - \delta\mathbb{1} - \bar{c}\right)$

where $\mathbb{1} = (\mathbb{1}_t)_{t=0}^T \in \mathcal{N}^2$ is defined as

$$\mathbb{1}_t = \begin{cases} (1,0) & \text{if } t = 0, \\ 0 & \text{if } t = 1, \ldots, T. \end{cases}$$

The *seller's regret indifference price* of the flow option c is defined as the lowest price δ that allows the investor to sell c without increasing his minimal regret, namely

$$\pi_{\mathrm{F}}^{ai}\left(c; \bar{c}\right) := \inf\left\{\delta \in \mathbb{R} \mid V\left(c - \delta\mathbb{1} - \bar{c}\right) \leq V\left(-\bar{c}\right)\right\}. \tag{3.51}$$

Similarly, in the situation when the investor is buying the flow option c, the investor receives the portfolio c_t at each time step $t = 0, \ldots, T$. Moreover, in return for receiving these portfolios, he delivers $\delta \in \mathbb{R}$ in cash at time step 0. By buying c, the investor's minimal regret becomes $V\left(-c + \delta\mathbb{1} - \bar{c}\right)$. The *buyer's regret indifference price* of c is defined as the highest price δ that allows the investor to buy the flow option c without increasing his minimal regret, namely

$$\pi_{\mathrm{F}}^{bi}\left(c; \bar{c}\right) := \sup\left\{\delta \in \mathbb{R} \mid V\left(-c + \delta\mathbb{1} - \bar{c}\right) \leq V\left(-\bar{c}\right)\right\}. \tag{3.52}$$

Notice that

$$\begin{aligned} \pi_{\mathrm{F}}^{bi}\left(c; \bar{c}\right) &= -\inf\left\{\delta \in \mathbb{R} \mid V\left(-c - \delta\mathbb{1} - \bar{c}\right) \leq V\left(-\bar{c}\right)\right\} \\ &= -\pi_{\mathrm{F}}^{ai}\left(-c; \bar{c}\right). \end{aligned} \tag{3.53}$$

Moreover, in the special case when $V\left(-\bar{c}\right) = \infty$, we have

$$\pi_{\mathrm{F}}^{ai}\left(c; \bar{c}\right) = \inf \mathbb{R} = -\infty, \tag{3.54}$$

$$\pi_{\mathrm{F}}^{bi}\left(c; \bar{c}\right) = \sup \mathbb{R} = \infty. \tag{3.55}$$

Remark 3.36. The regret indifference prices $\pi_{\mathrm{F}}^{ai}\left(c; \bar{c}\right)$ and $\pi_{\mathrm{F}}^{bi}\left(c; \bar{c}\right)$ of c depend on the investor's endowment $\bar{c}$. In addition, the regret indifference prices depend on V, the value function of the optimisation problem (3.8). Clearly, the regret indifference prices depend on the choice of regret functions $(v_t)_{t=0}^T$.

The regret indifference prices defined in (3.51)-(3.52) above are similar to the *indifference swap rates* defined in Pennanen (2014). Pennanen (2014) concerns the value of cash flows instead of flow options. However, in the case when c and $\bar{c}$ are cash flows (i.e. $c_t = (c_t^b, 0)$ and $\bar{c}_t = (\bar{c}_t^b, 0)$ for all $t = 0, \ldots, T$), regret indifference prices are special examples of indifference swap rates.

Indifference prices based on utility maximisation has been studied widely; see Davis, Panas & Zariphopoulou (1993), Rouge & El Karoui (2000), Musiela & Zariphopoulou (2004), Hugonnier, Kramkov & Schachermayer (2005), Mania & Schweizer (2005), Cetin & Rogers (2007), Carmona (2009), Benth, Groth & Lindberg (2010), Quek (2012). Regret indifference prices are similar to but more general than the utility indifference prices. This is mainly because the investor's preference towards risks is allowed to different at different time steps. Moreover, regret indifference prices depend on investor's endowment which extends initial wealth used in utility maximisation problems. In addition, regret indifference pricing can be used to evaluate the value of flow options which extends cash flows and European options.

The example below shows that superhedging pricing defined in (2.24) and (2.25) is a special case of regret indifference pricing.

Example 3.37. For every $t = 0, \dots, T$, let $\bar{c}_t = 0$ and $v_t = \delta_{(-\infty,0]}$. From Example 3.11, we have for all $u \in \mathcal{N}^2$ that

$$V(u) = \begin{cases} 0 & \text{if } \exists y \in \Psi : \phi_t(\Delta y_t + u_t) \leq 0 \, \forall t = 0, \dots, T, \\ \infty & \text{otherwise.} \end{cases}$$

Observe that $V(0) = 0$.

Fix any $c = (c_t)_{t=0}^T \in \mathcal{N}^2$ and $\delta \in \mathbb{R}$. Observe that for every $(y_t)_{t=-1}^T \in \Psi$ and $(y_t^*)_{t=-1}^T \in \mathcal{N}^{2\prime}$, if $y_{-1}^* = (\delta, 0)$ and $y_t^* = y_t$ for all $t = 0, \dots, T$, then $y_{-1} = 0$ gives

$$\phi_t(\Delta y_0 + c_0 - \delta \mathbb{1}_0) = \phi_0(y_0 + c_0 - (\delta, 0)) = \phi_0(\Delta y_0^* + c_0)$$

and

$$\phi_t(\Delta y_t + c_t - \delta \mathbb{1}_t) = \phi_t(\Delta y_t + c_t) = \phi_t(\Delta y_t^* + c_t) \text{ for all } t = 1, \dots, T.$$

This implies that there exists $(y_t)_{t=-1}^T \in \Psi$ such that

$$\phi_t(\Delta y_t + c_t - \delta \mathbb{1}_t) \leq 0 \text{ for all } t = 0, \dots, T$$

if and only if there exists $(y_t^*)_{t=-1}^T \in \mathcal{N}^{2\prime}$ such that

$$y_{-1}^* = (\delta, 0), \, y_T^* = 0, \, \phi_t(\Delta y_t^* + c_t) \leq 0 \text{ for all } t = 0, \dots, T.$$

Thus

$$\begin{aligned} \pi_{\mathrm{F}}^{ai}(c; 0) &= \inf \{\delta \in \mathbb{R} \mid V(c - \delta \mathbb{1}) \leq V(0)\} \\ &= \inf \{\delta \in \mathbb{R} \mid V(c - \delta \mathbb{1}) \leq 0\} \\ &= \inf \left\{\delta \in \mathbb{R} \mid (y_t)_{t=-1}^T \in \Psi, \phi_t(\Delta y_t + c_t - \delta \mathbb{1}_t) \leq 0 \, \forall t = 0, \dots, T\right\} \end{aligned}$$

$$= \inf \Big\{ \delta \in \mathbb{R} \mid (y_t^*)_{t=-1}^T \in \mathcal{N}^{2\prime}, y_{-1}^* = (\delta, 0), y_T^* = 0,$$
$$\phi_t(\Delta y_t^* + c_t) \leq 0 \, \forall t = 0, \ldots, T \Big\},$$

and hence $\pi_{\mathrm{F}}^{ai}(c; 0) = \pi_{\mathrm{F}}^{a}(c)$ by (2.24). In addition, we have

$$\pi_{\mathrm{F}}^{bi}(c; 0) = -\pi_{\mathrm{F}}^{ai}(-c; 0) = -\pi_{\mathrm{F}}^{a}(-c) = \pi_{\mathrm{F}}^{b}(c).$$

Under other types of regret functions, it is possible that $\pi_{\mathrm{F}}^{ai}(c;0) < \pi_{\mathrm{F}}^{a}(c)$ and $\pi_{\mathrm{F}}^{bi}(c;0) > \pi_{\mathrm{F}}^{b}(c)$; see Table 5.1 in Example 5.10.

The lemma below will be useful for establishing Theorem 3.39.

Lemma 3.38. *The following two claims hold true.*

1. *For every $\delta \leq 0$, we have $0 \in A_{\delta \mathbb{1}}$.*

2. *For every $u \in \mathcal{N}^2$, we have $0 \in A_{u - \pi_{\mathrm{F}}^{a}(u)\mathbb{1}}$.*

Proof. For all $\delta \leq 0$, we have

$$\mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T \delta \mathbb{1}_t \right] = \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot (\delta, 0) \right] = \delta \leq 0 \text{ for all } (\mathbb{Q}, S) \in \bar{\mathcal{P}},$$

and hence $0 \in A_{\delta \mathbb{1}}$ by (3.20). Thus the first claim holds true. Fix any $u \in \mathcal{N}^2$. Notice from (2.26) and Theorem 2.14 that

$$\pi_{\mathrm{F}}^{a}(u) = \max_{(\mathbb{Q},S) \in \bar{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T u_t \right] = \sup_{(\mathbb{Q},S) \in \bar{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T u_t \right].$$

For any $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$, we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T (u_t - \pi_{\mathrm{F}}^{a}(u) \mathbb{1}_t) \right] &= \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \left(\textstyle\sum_{t=0}^T u_t - (\pi_{\mathrm{F}}^{a}(u), 0) \right) \right] \\ &= \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T u_t \right] - \pi_{\mathrm{F}}^{a}(u) \leq 0. \end{aligned}$$

Thus $0 \in A_{u - \pi_{\mathrm{F}}^{a}(u)\mathbb{1}}$ by (3.20) again, which establishes the second claim. □

The theorem below says that the indifference price for the seller is not going to be higher than seller's arbitrage price, and that the indifference price for the buyer is not going to be lower than buyer's arbitrage price. Moreover, if the indifference price for the seller of the flow option $0 \in \mathcal{N}^2$ is zero (it does not always hold true, see Example 3.43 below), then the seller's indifference price will not be lower than buyer's indifference price.

Theorem 3.39. *Under the assumption that the robust no-arbitrage condition holds true, for every $\bar{c}, c \in \mathcal{N}^2$, we always have*

$$\pi_{\mathrm{F}}^{ai}(c;\bar{c}) \leq \pi_{\mathrm{F}}^{a}(c), \qquad \pi_{\mathrm{F}}^{bi}(c;\bar{c}) \geq \pi_{\mathrm{F}}^{b}(c).$$

Additionally, if $\pi_{\mathrm{F}}^{ai}(0;\bar{c}) = 0$, then $c' \mapsto \pi_{\mathrm{F}}^{ai}(c';\bar{c})$ is real-valued and convex on $\mathcal{N}^2$, and moreover

$$\pi_{\mathrm{F}}^{b}(c) \leq \pi_{\mathrm{F}}^{bi}(c;\bar{c}) \leq \pi_{\mathrm{F}}^{ai}(c;\bar{c}) \leq \pi_{\mathrm{F}}^{a}(c).$$

Proof. Notice that $0 \in A_{c-\pi_{\mathrm{F}}^{a}(c)\mathbb{1}}$ (Lemma 3.38.2). Then Lemma 3.19 gives

$$V\left(c - \pi_{\mathrm{F}}^{a}(c)\,\mathbb{1} - \bar{c}\right) \leq V(-\bar{c}).$$

Thus, the definition of $\pi_{\mathrm{F}}^{ai}(c;\bar{c})$ in (3.51) implies $\pi_{\mathrm{F}}^{ai}(c;\bar{c}) \leq \pi_{\mathrm{F}}^{a}(c)$. Since c is arbitrary, we have $\pi_{\mathrm{F}}^{ai}(c';\bar{c}) \leq \pi_{\mathrm{F}}^{a}(c')$ for every $c' \in \mathcal{N}^2$. This implies that

$$\pi_{\mathrm{F}}^{ai}(-c;\bar{c}) \leq \pi_{\mathrm{F}}^{a}(-c),$$

and hence

$$\pi_{\mathrm{F}}^{bi}(c;\bar{c}) = -\pi_{\mathrm{F}}^{ai}(-c;\bar{c}) \geq -\pi_{\mathrm{F}}^{a}(-c) = \pi_{\mathrm{F}}^{b}(c)$$

which completes the proof of the first claim.

Suppose that $\pi_{\mathrm{F}}^{ai}(0;\bar{c}) = 0$ for the remainder of the proof. Firstly, we are going to show that $c' \mapsto \pi_{\mathrm{F}}^{ai}(c';\bar{c})$ is real-valued on $\mathcal{N}^2$. Observe from (2.26) and Theorem 2.14 that $|\pi_{\mathrm{F}}^{a}(c)| < \infty$. This means $\pi_{\mathrm{F}}^{ai}(c;\bar{c}) \leq \pi_{\mathrm{F}}^{a}(c) < \infty$. To prove $\left|\pi_{\mathrm{F}}^{ai}(c;\bar{c})\right| < \infty$, it is sufficient to show that $\pi_{\mathrm{F}}^{ai}(c;\bar{c}) > -\infty$. From $\pi_{\mathrm{F}}^{ai}(0;\bar{c}) = 0$ and (3.51), we have

$$\inf\left\{\delta \in \mathbb{R} \mid V(-\delta\mathbb{1} - \bar{c}) \leq V(-\bar{c})\right\} = 0.$$

This implies that

$$V\left(-\delta'\mathbb{1} - \bar{c}\right) > V(-\bar{c}) \text{ for all } \delta' < 0.$$

Let $\delta' < 0$, and let $\delta^* := \delta' - \pi_{\mathrm{F}}^{a}(-c)$. Notice that

$$c - \delta^*\mathbb{1} - \bar{c} + \left[-c + \left(\delta^* - \delta'\right)\mathbb{1}\right] = -\delta'\mathbb{1} - \bar{c}.$$

Moreover, Lemma 3.38.2 implies $0 \in A_{-c-\pi_{\mathrm{F}}^{a}(-c)\mathbb{1}} = A_{-c+(\delta^*-\delta')\mathbb{1}}$. Combining this with Lemma 3.19, it follows that

$$V\left(c-\delta^{*}\mathbb{1}-\bar{c}\right)\geq V\left(-\delta'\mathbb{1}-\bar{c}\right)>V\left(-\bar{c}\right).$$

For all $\delta<\delta^{*}$, it follows from $0\in A_{(\delta-\delta^{*})\mathbb{1}}$ (see Lemma 3.38.1) and Lemma 3.19 that

$$V\left(c-\delta\mathbb{1}-\bar{c}\right)\geq V\left(c-\delta^{*}\mathbb{1}-\bar{c}\right)>V\left(-\bar{c}\right).$$

Therefore, we have

$$\pi^{ai}_{\mathrm{F}}(c;\bar{c})=\inf\left\{\delta\in\mathbb{R}\mid V\left(c-\delta\mathbb{1}-\bar{c}\right)\leq V\left(-\bar{c}\right)\right\}\geq\delta^{*}>-\infty.$$

We can conclude that $\left|\pi^{ai}_{\mathrm{F}}(c;\bar{c})\right|<\infty$. Notice that c is arbitrary, and this implies that the function $c'\mapsto\pi^{ai}_{\mathrm{F}}\left(c';\bar{c}\right)$ is real-valued.

We are going to prove the convexity of $c'\mapsto\pi^{ai}_{\mathrm{F}}\left(c';\bar{c}\right)$ as follows. Firstly, we define

$$C:=\left\{x\in\mathcal{N}^{2}\middle|\,V\left(x-\bar{c}\right)\leq V\left(-\bar{c}\right)\right\}.$$

Using the convexity of V (see the comments following (3.15)), we have for any $x,y\in C$ and $\gamma\in(0,1)$ that

$$\begin{aligned}V(\gamma x+(1-\gamma)y-\bar{c})&\leq\gamma V\left(x-\bar{c}\right)+\left(1-\gamma\right)V\left(y-\bar{c}\right)\\&\leq\gamma V\left(-\bar{c}\right)+\left(1-\gamma\right)V\left(-\bar{c}\right)\\&=V\left(-\bar{c}\right).\end{aligned}$$

This implies that $\gamma x+(1-\gamma)y\in C$. Thus C is convex. Now, fix any $c^{1},c^{2}\in\mathcal{N}^{2}$ and $\mu\in(0,1)$. Then it follows from (3.51) that

$$\begin{aligned}&\mu\pi^{ai}_{\mathrm{F}}(c^{1};\bar{c})+(1-\mu)\pi^{ai}_{\mathrm{F}}(c^{2};\bar{c})\\&\qquad=\mu\inf\left\{\delta^{1}\middle|\,c^{1}-\delta^{1}\mathbb{1}\in C\right\}+(1-\mu)\inf\left\{\delta^{2}\middle|\,c^{2}-\delta^{2}\mathbb{1}\in C\right\}\\&\qquad=\inf\left\{\mu\delta^{1}+(1-\mu)\delta^{2}\middle|\,c^{1}-\delta^{1}\mathbb{1}\in C,\,c^{2}-\delta^{2}\mathbb{1}\in C\right\}.\end{aligned}$$

Observe that, for any $\delta^{1},\delta^{2}\in\mathbb{R}$ such that $c^{1}-\delta^{1}\mathbb{1}\in C$ and $c^{2}-\delta^{2}\mathbb{1}\in C$, by taking $\delta=\mu\delta^{1}+(1-\mu)\delta^{2}$, it follows that

$$\mu c^{1}+(1-\mu)c^{2}-\delta\mathbb{1}=\mu\left(c^{1}-\delta^{1}\mathbb{1}\right)+(1-\mu)\left(c^{2}-\delta^{2}\mathbb{1}\right)\in C$$

by the convexity of C. This implies

$$\mu\pi^{ai}_{\mathrm{F}}(c^{1};\bar{c})+(1-\mu)\pi^{ai}_{\mathrm{F}}(c^{2};\bar{c})$$

$$
\begin{aligned}
&\geq \inf\left\{\delta\in\mathbb{R}\,\middle|\,\mu c^1+(1-\mu)c^2-\delta\mathbb{1}\in C\right\}\\
&=\inf\left\{\delta\in\mathbb{R}\,\middle|\,V\left(\mu c^1+(1-\mu)c^2-\delta\mathbb{1}-\bar{c}\right)\leq V\left(-\bar{c}\right)\right\}\\
&=\pi_{\mathrm{F}}^{ai}(\mu c^1+(1-\mu)c^2;\bar{c}).
\end{aligned}
$$

This proves the convexity of $c'\mapsto\pi_{\mathrm{F}}^{ai}\left(c';\bar{c}\right)$. Thus $c'\mapsto\pi_{\mathrm{F}}^{ai}\left(c';\bar{c}\right)$ is real-valued and convex on $\mathcal{N}^2$.

Since $\pi_{\mathrm{F}}^{ai}\left(0;\bar{c}\right)=0$ and $c'\mapsto\pi_{\mathrm{F}}^{ai}\left(c';\bar{c}\right)$ is convex on $\mathcal{N}^2$, we have

$$
0=\pi_{\mathrm{F}}^{ai}(\tfrac{1}{2}c+\tfrac{1}{2}\left(-c\right);\bar{c})\leq\tfrac{1}{2}\pi_{\mathrm{F}}^{ai}\left(c;\bar{c}\right)+\tfrac{1}{2}\pi_{\mathrm{F}}^{ai}\left(-c;\bar{c}\right).
$$

Thus

$$
\pi_{\mathrm{F}}^{ai}\left(c;\bar{c}\right)\geq-\pi_{\mathrm{F}}^{ai}\left(-c;\bar{c}\right)=\pi_{\mathrm{F}}^{bi}\left(c;\bar{c}\right)
$$

which completes the proof. □

The result below provides a sufficient condition to ensure $\pi_{\mathrm{F}}^{ai}\left(0;\bar{c}\right)=0$. This sufficient condition requires $V(-\bar{c})<\infty$. Clearly, when $V(-\bar{c})=\infty$, we have $\pi_{\mathrm{F}}^{ai}\left(0;\bar{c}\right)=-\infty$. Moreover, it also requires that there exists some $t^*=0,\ldots,T$ and $\nu\in\Omega_{t^*}$ such that $v_{t^*}^{\nu}$ is increasing on its effective domain $\operatorname{dom}v_{t^*}^{\nu}$, in other words,

$$
v_{t^*}^{\nu}(x)<v_{t^*}^{\nu}(x')\text{ for all }x,x'\in\operatorname{dom}v_{t^*}^{\nu}\text{ such that }x<x'.
$$

There are many increasing regret functions. For example, exponential regret functions and power regret functions are always increasing on their effective domains; see Examples 3.4.1-3.4.2.

Proposition 3.40. *Suppose that $\bar{c}\in\mathcal{N}^2$ such that $V\left(-\bar{c}\right)<\infty$, and that there exists some $t^*=0,\ldots,T$ and $\nu\in\Omega_{t^*}$ such that $x\mapsto v_{t^*}^{\nu}(x)$ is increasing on $\operatorname{dom}v_{t^*}^{\nu}$. Then, for any $c\in\mathcal{N}^2$ and $\delta\in\mathbb{R}$ such that*

$$
V\left(c-\delta\mathbb{1}-\bar{c}\right)=V\left(-\bar{c}\right), \tag{3.56}
$$

we have $\pi_{\mathrm{F}}^{ai}\left(c;\bar{c}\right)=\delta$. In particular, we have $\pi_{\mathrm{F}}^{ai}\left(0;\bar{c}\right)=0$.

Proof. It is sufficient to show that, for any $\epsilon>0$ and $u\in\mathcal{N}^2$ such that $V(u)<\infty$, we have

$$
V(u+\epsilon\mathbb{1})>V(u). \tag{3.57}
$$

Then, for any $c\in\mathcal{N}^2$ and $\delta\in\mathbb{R}$ such that (3.56) holds true, we have for every $\delta'\in(-\infty,\delta)$ that

$$
V\left(c-\delta\mathbb{1}-\bar{c}+(\delta-\delta')\mathbb{1}\right)>V\left(c-\delta\mathbb{1}-\bar{c}\right)=V\left(-\bar{c}\right),
$$

in other words,

$$V\left(c - \delta' \mathbb{1} - \bar{c}\right) > V\left(-\bar{c}\right).$$

Combining this with the definition of $\pi_{\mathrm{F}}^{ai}\left(c;\bar{c}\right)$ in (3.51), we have $\pi_{\mathrm{F}}^{ai}\left(c;\bar{c}\right) \geq \delta$. Moreover, it follows from (3.51) and (3.56) that $\pi_{\mathrm{F}}^{ai}\left(c;\bar{c}\right) \leq \delta$. Therefore, we must have $\pi_{\mathrm{F}}^{ai}\left(c;\bar{c}\right) = \delta$. In particular, the condition (3.56) is satisfied for $c = 0$ and $\delta = 0$, and this means $\pi_{\mathrm{F}}^{ai}\left(0;\bar{c}\right) = 0$.

Suppose now that $x \mapsto v_{t^*}^{\nu}(x)$ is increasing for some $t^* = 0,\ldots,T$ and $\nu \in \Omega_{t^*}$. Fix any $\epsilon > 0$ and $u = (u_t)_{t=0}^T \in \mathcal{N}^2$ such that $V(u) < \infty$. We are going to show that (3.57) holds true. Consider the following two cases. If $V(u+\epsilon\mathbb{1}) = \infty$, then $V(u) < \infty$ gives (3.57). Suppose that $V(u+\epsilon\mathbb{1}) < \infty$ for the remainder of this proof. From Corollary 3.18 and (3.21), there exists $\hat{x} = (\hat{x}_t)_{t=0}^T \in \mathcal{N}$ such that $\hat{x} \in A_{u+\epsilon\mathbb{1}}$ and

$$\sum_{t=0}^T \mathbb{E}\left[v_t(\hat{x}_t)\right] = \inf_{x\in A_{u+\epsilon\mathbb{1}}} \sum_{t=0}^T \mathbb{E}\left[v_t\left(x_t\right)\right] = V(u+\epsilon\mathbb{1});$$

see (3.20) for the definition of $A_{u'}$ for all $u' \in \mathcal{N}^2$. Define $y = (y_t)_{t=0}^T \in \mathcal{N}$ as

$$y_t = \begin{cases} \hat{x}_{t^*} - \epsilon & \text{on } \nu \text{ if } t = t^* \\ \hat{x}_t & \text{otherwise.} \end{cases} \tag{3.58}$$

Fix any $t = 0,\ldots,T$. Notice that $v_t(\hat{x}_t) > -\infty$ because regret functions are always bounded from below. In addition, it follows from $V(u+\epsilon\mathbb{1}) < \infty$ that $v_t(\hat{x}_t) < \infty$. Thus $v_t(\hat{x}_t)$ is a finite value. We are going to present $\mathbb{E}[v_t(\hat{x}_t) - v_t(y_t)]$ by considering the following two situations. In the situation when $t \neq t^*$, the definition of y_t in (3.58) gives

$$\mathbb{E}\left[v_t(\hat{x}_t) - v_t(y_t)\right] = \mathbb{E}\left[v_t(\hat{x}_t) - v_t(\hat{x}_t)\right] = 0.$$

Moreover, in the situation when $t = t^*$, it follows that

$$\begin{aligned} &\mathbb{E}\left[v_{t^*}(\hat{x}_{t^*}) - v_{t^*}(y_{t^*})\right] \\ &\quad= \mathbb{E}\left[\left(v_{t^*}(\hat{x}_{t^*}) - v_{t^*}(y_{t^*})\right)\mathbf{1}_\nu\right] + \mathbb{E}\left[\left(v_{t^*}(\hat{x}_{t^*}) - v_{t^*}(y_{t^*})\right)\mathbf{1}_{\Omega\setminus\nu}\right] \\ &\quad= \mathbb{P}(\nu)\left[v_{t^*}^{\nu}(\hat{x}_{t^*}(\nu)) - v_{t^*}^{\nu}(\hat{x}_{t^*}(\nu) - \epsilon)\right] > 0 \end{aligned}$$

because $\mathbb{P}(\nu) > 0$ and $v_{t^*}^{\nu}$ is increasing. Thus

$$V(u+\epsilon\mathbb{1}) - \sum_{t=0}^T \mathbb{E}\left[v_t(y_t)\right] = \sum_{t=0}^T \mathbb{E}\left[v_t(\hat{x}_t)\right] - \sum_{t=0}^T \mathbb{E}\left[v_t(y_t)\right] > 0. \tag{3.59}$$

Fix any $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$. Notice that

$$\epsilon = \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot (\epsilon, 0)\right] = \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^{T} \epsilon \mathbb{1}_t\right]$$

and hence

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^{T} u_t\right] =& \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^{T} u_t\right] + \epsilon - \epsilon \\ =& \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^{T} (u_t + \epsilon \mathbb{1}_t)\right] - \epsilon.\end{aligned}$$

Moreover, observe from (3.58) that

$$\mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^{T} (y_t, 0)\right] = \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^{T} (\hat{x}_t, 0)\right] - \mathbb{Q}(\nu)\epsilon.$$

Therefore, we have

$$\begin{aligned}&\mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^{T} (u_t - (y_t, 0))\right] \\ &\qquad = \mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot \textstyle\sum_{t=0}^{T} (u_t + \epsilon \mathbb{1}_t - (\hat{x}_t, 0))\right] - (1 - \mathbb{Q}(\nu))\epsilon \leq 0\end{aligned}$$

because $\hat{x} \in A_{u+\epsilon \mathbb{1}}$ and $-(1 - \mathbb{Q}(\nu))\epsilon \leq 0$. This means $y \in A_u$. Then it follows from (3.21) that

$$V(u) = \inf_{x \in A_u} \sum_{t=0}^{T} \mathbb{E}\left[v_t(x_t)\right] \leq \sum_{t=0}^{T} \mathbb{E}\left[v_t(y_t)\right].$$

This implies

$$V(u + \epsilon \mathbb{1}) - V(u) \geq V(u + \epsilon \mathbb{1}) - \sum_{t=0}^{T} \mathbb{E}\left[v_t(y_t)\right].$$

Observe from (3.59) that (3.57) holds true. This completes the proof. □

The following example shows that sometimes $\pi_{\mathrm{F}}^{ai}(0; 0) = 0$ even if there exists no $t^* = 0, \ldots, T$ and $\nu \in \Omega_{t^*}$ such that $v_{t^*}^{\nu}$ is increasing on its effective domain $\operatorname{dom} v_{t^*}^{\nu}$

Example 3.41. Let $v_t = \delta_{(-\infty, 0]}$ for all $t = 0, \ldots, T$. Then there does not exist $t^* = 0, \ldots, T$ and $\nu \in \Omega_{t^*}$ such that the function $v_{t^*}^{\nu}$ is increasing on $\operatorname{dom} v_{t^*}^{\nu}$. Thus the assumption in Proposition 3.40 is not satisfied. However, by Example 3.37 and Theorem 2.14, we have $\pi_{\mathrm{F}}^{ai}(0; 0) = \pi_{\mathrm{F}}^{a}(0) = 0$.

The following example shows that, under the regret functions $(v_t)_{t=0}^{T}$ defined in Example 3.26, we have $\pi_{\mathrm{F}}^{ai}(0; \bar{c}) = 0$ for all $\bar{c} \in \mathcal{N}^2$.

Example 3.42. Consider $(v_t)_{t=0}^T$ defined in Example 3.26, and fix any $\bar{c} \in \mathcal{N}^2$.

Firstly, we have for each $\nu \in \Omega_T$ that

$$v_T^\nu(x) = e^{\alpha_T x} - 1 \text{ for all } x \in \mathbb{R}$$

where $\alpha_T > 0$ is independent of ν, and moreover the regret function v_T^ν is increasing. Observe that $v_T(x) < \infty$ for all $x \in \mathcal{L}_T$.

Secondly, define $(x_t)_{t=0}^T \in \mathcal{N}$ as $x_t := 0$ for $t = 0, \dots, T-1$ and

$$x_T := \max_{(\mathbb{Q},S) \in \bar{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T (-\bar{c}_t) \right];$$

from Theorem 2.14 the maximum exists. Then we have for all $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$ that

$$\mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T (-\bar{c}_t - (x_t, 0)) \right] = \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^T (-\bar{c}_t) \right] - x_T \le 0$$

and hence $(x_t)_{t=0}^T \in A_{-\bar{c}}$. Moreover, we have $v_t(x_t) = v_t(0) = 0$ for every $t = 0, \dots, T-1$, which implies

$$\sum_{t=0}^T \mathbb{E}\left[v_t(x_t)\right] = \mathbb{E}\left[v_T(x_T)\right] < \infty.$$

Thus, it follows from (3.21) that $V(-\bar{c}) < \infty$. Then $\pi_{\mathrm{F}}^{ai}(0; \bar{c}) = 0$ by Proposition 3.40.

The example below shows that, sometimes, whether $\pi_{\mathrm{F}}^{ai}(0; \bar{c})$ is 0 or not depends on the choice of $\bar{c}$. Moreover, when $\pi_{\mathrm{F}}^{ai}(0; \bar{c}) \neq 0$, it is possible that the buyer's indifference is greater than the seller's indifference price.

Example 3.43. Suppose that $v_t = \delta_{(-\infty,0]}$ for all $t = 0, \dots, T-1$ and

$$v_T(x) = \begin{cases} -1 + \frac{1}{1-x} & \text{if } x < 1, \\ \infty & \text{if } x \ge 1. \end{cases}$$

Observe that v_T^ν is a power regret function for every $\nu \in \Omega_T$; see Example 3.4.2. We are going to show that $\pi_{\mathrm{F}}^{ai}(0; \bar{c})$ depends on the choice of endowment $\bar{c}$ by considering the following two situations.

Suppose that the investor's endowment $\bar{c} = (\bar{c}_t)_{t=0}^T \in \mathcal{N}^2$ is

$$\bar{c}_t = \begin{cases} 0 & \text{if } t = 0, \dots, T-1, \\ -(1, 0) & \text{if } t = T. \end{cases}$$

Then (3.9) gives

$$\begin{aligned} V(-\bar{c}) &= \inf_{y\in\Psi} \textstyle\sum_{t=0}^{T} \mathbb{E}\left[v_t(\phi_t(\Delta y_t - \bar{c}_t))\right] \\ &= \inf_{y\in\Psi} \left\{ \textstyle\sum_{t=0}^{T-1} \mathbb{E}\left[v_t(\phi_t(\Delta y_t))\right] + \mathbb{E}\left[v_T(\phi_T(\Delta y_t + (1,0)))\right] \right\} \\ &= \inf_{y\in\Psi} \left\{ \textstyle\sum_{t=0}^{T-1} \mathbb{E}\left[v_t(\phi_t(\Delta y_t))\right] + \mathbb{E}\left[v_T(\phi_T(\Delta y_t) + 1)\right] \right\}. \end{aligned}$$

By the construction of $(v_t)_{t=0}^{T}$, the value $V(-\bar{c}) < \infty$ if and only if

$$\{ y \in \Psi |\, \phi_T(\Delta y_T) < 0,\, \phi_t(\Delta y_t) \leq 0 \text{ for all } t = 0, \ldots, T-1 \} \tag{3.60}$$

is not empty. Suppose by contradiction that there exists y^* that belongs to the set (3.60). Then (2.8) gives $y^* \in \Phi \cap \Psi$. However, Proposition 2.7 implies that under the no-arbitrage condition we have $\phi_T(\Delta y_T^*) = 0$ which violates the condition $\phi_T(\Delta y_T^*) < 0$ in (3.60). Therefore, the set (3.60) is empty and hence $V(-\bar{c}) = \infty$. Fix any $c \in \mathcal{N}^2$. Then $\pi_{\mathrm{F}}^{ai}(c;\bar{c}) = -\infty$ and $\pi_{\mathrm{F}}^{bi}(c;\bar{c}) = \infty$ by (3.54) and (3.55) respectively. Thus $\pi_{\mathrm{F}}^{bi}(c;\bar{c}) > \pi_{\mathrm{F}}^{ai}(c;\bar{c})$. In particular, the indifference price for the seller $\pi_{\mathrm{F}}^{ai}(0;\bar{c}) = -\infty$ is not zero. Therefore, it is possible that the buyer's indifference is greater than the seller's indifference price when $\pi_{\mathrm{F}}^{ai}(0;\bar{c}) \neq 0$.

Suppose now that the investor's endowment $\bar{c} = (\bar{c}_t)_{t=0}^{T} \in \mathcal{N}^2$ is given by $\bar{c}_t = 0$ for all $t = 0, \ldots, T$. Define $y' = (y'_t)_{t=-1}^{T} \in \Psi$ by $y'_t := 0$ for all $t = -1, \ldots, T$. Observe from (3.9) that

$$V(0) = \inf_{y\in\Psi} \sum_{t=0}^{T} \mathbb{E}\left[v_t(\phi_t(\Delta y_t))\right] \leq \sum_{t=0}^{T} \mathbb{E}\left[v_t(\phi_t(\Delta y'_t))\right] = \sum_{t=0}^{T} \mathbb{E}\left[v_t(\phi_t(0))\right] = 0,$$

and hence $V(-\bar{c}) = V(0) \leq 0$. Moreover, the function v_T^{ν} is increasing on its effective domain $\operatorname{dom} v_T^{\nu}$ for every $\nu \in \Omega_T$. Then $\pi_{\mathrm{F}}^{ai}(0;\bar{c}) = 0$ by Proposition 3.40.

Chapter 4

Extending the convex hull

In this chapter, we will present a number of technical results that will be applied in Chapter 5 for studying the dual problem (3.35) under the exponential regret functions introduced in Example 3.26. All the results established in this chapter are technical rather than connected with any particular financial model. Moreover, these results do not rely on any result from previous chapters. In Section 4.1, we will introduce a minimisation problem for which the value function is formulated as an extended convex hull of a collection of convex functions. In Theorem 4.3, we will show that the value function is convex. The focus in Section 4.2 will be on the proof of the existence of a solution to this minimisation problem. With the help of a number of technical results in Rockafellar (1997), we will establish the main result in Theorem 4.13 which proves the existence of a solution and the continuity of the value function. An example of the minimisation problem with an entropy type function will be presented in Section 4.3. In this example, we will provide a method to construct the solutions to this problem by considering all different cases of the values of given parameters.

4.1 Problem formulation

In this section, we will first introduce an optimisation problem. The value function of this problem can be regarded as an extended convex hull of a collection of convex functions; see Remark 4.1. Then Theorem 4.3 shows that the value function is convex, and its effective domain is provided in (4.5).

Let $m \geq 2$ be an integer. For every $i = 1, \dots, m$, let f_i be an $\mathbb{R} \cup \{\infty\}$-valued proper convex function on $\mathbb{R}$ that is bounded from below. Notice that the epigraph $\operatorname{epi} f_i \neq \emptyset$ is a convex set because f_i is proper and convex. Moreover, we assume that $\operatorname{epi} f_i$ is closed, and that the recession cone of $\operatorname{epi} f_i$

satisfies

$$(\operatorname{epi} f_i)^\infty = \{\,(0,b) \,|\, b \geq 0\}. \tag{4.1}$$

For example, if the effective domain $\operatorname{dom} f_i$ is bounded, then (4.1) holds true. The definitions of epigraph, recession cone and effective domain can be found in Appendix A.1.

For each $i = 1,\dots,m$, let both g_i^1 and g_i^2 be $\mathbb{R} \cup \{\infty\}$-valued convex functions on $\mathbb{R}$ that are bounded from below. We assume that

$$[0,1] \subseteq \operatorname{dom} g_i^1$$

and that

$$\operatorname{cl}\,\{\lambda x \,|\lambda \in [0,1], x \in \operatorname{dom} f_i\,\} \subseteq \operatorname{dom} g_i^2,$$

where $\operatorname{cl} A$ is the closure of a given set A. Moreover, the functions g_i^1 and g_i^2 are assumed to be continuous on $\operatorname{dom} g_i^1$ and $\operatorname{dom} g_i^2$ respectively, and moreover

$$g_i^1\,(0) = g_i^2(0) = 0. \tag{4.2}$$

Notice that the epigraphs $\operatorname{epi} g_i^1$ and $\operatorname{epi} g_i^2$ are closed and convex.

For all $x \in \mathbb{R}$, let

$$\begin{aligned} f(x) := \inf \Bigg\{ \sum_{i=1}^m \Big(\lambda_i f_i(x_i) + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i)\Big) \Bigg| \\ \lambda_i \in [0,1], x_i \in \operatorname{dom} f_i \;\forall i = 1,\dots,m, \sum_{i=1}^m \lambda_i = 1, \sum_{i=1}^m \lambda_i x_i = x \Bigg\}. \end{aligned} \tag{4.3}$$

The value $f(x)$ is defined as the optimal value of a minimisation problem with parameter x. In this problem, the control variables $\lambda_1,\dots,\lambda_m \in [0,1]$ are weights, and the control variables $x_1,\dots,x_m$ take their values in the intervals $\operatorname{dom} f_1,\dots,\operatorname{dom} f_m$. The infimum in (4.3) is attained if and only if there exists a solution to this problem (i.e. there exists $(\lambda_1, x_1,\dots,\lambda_m, x_m)$ such that the constraints in (4.3) are satisfied and $\sum_{i=1}^m (\lambda_i f_i(x_i) + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i)) = f(x)$).

Remark 4.1. In the case when $g_i^1 = g_i^2 = 0$ for all $i = 1,\dots,m$, we have for all $x \in \mathbb{R}$ that

$$\begin{aligned} f(x) = \inf \Bigg\{ \sum_{i=1}^m \lambda_i f_i(x_i) \Bigg| \\ \lambda_i \in [0,1], x_i \in \operatorname{dom} f_i \;\forall i = 1,\dots,m, \sum_{i=1}^m \lambda_i = 1, \sum_{i=1}^m \lambda_i x_i = x \Bigg\}. \end{aligned}$$

Then f is reduced to the *convex hull* of $f_1,\dots,f_m$. This is the greatest convex function such that $f \leq f_i$ for all $i = 1, \ldots, m$ (Rockafellar 1997, Theorem 5.6). This means that f defined in (4.3) is an extension of the convex hull of $f_1, \ldots, f_m$.

Remark 4.2. The minimisation problem in (4.3) is slightly more general than the later problems in (5.42), (5.60), and (5.76) considered in Chapter 5. These three problems will correspond to special examples of the problem in (4.3) for $g_1^2 = \cdots = g_m^2 = 0$.

The functions $f_1, g_1^1, g_1^2, \ldots, f_m, g_m^1, g_m^2$ are bounded from below. Then it follows from the definition of f in (4.3) that

$$f > -\infty \text{ on } \mathbb{R}. \tag{4.4}$$

Thus, the function f is of the form $\mathbb{R} \to \mathbb{R} \cup \{\infty\}$. The theorem below shows that f is convex; the convexity of f only relies on the convexity of the functions $f_1, g_1^1, g_1^2, \ldots, f_m, g_m^1, g_m^2$. In addition, this theorem also shows that the effective domain of f is given by $\mathrm{co}\left(\cup_{i=1}^m \mathrm{dom}\, f_i\right)$, where $\mathrm{co}\,(A)$ is the convex hull of any given set A.

Theorem 4.3. *The function f defined in* (4.3) *is $\mathbb{R} \cup \{\infty\}$-valued and convex on $\mathbb{R}$. Moreover, its effective domain is*

$$\mathrm{dom}\, f = \mathrm{co}\left(\bigcup_{i=1}^m \mathrm{dom}\, f_i\right). \tag{4.5}$$

Proof. Firstly, the function f is $\mathbb{R} \cup \{\infty\}$-valued; see (4.4). Secondly, we shall establish the convexity of f. Fix any $x, y \in \mathbb{R}$ and $a \in (0, 1)$. To show that f is convex, we are going to prove

$$f\left(ax + (1-a)\, y\right) \leq af\left(x\right) + (1-a)\, f\left(y\right). \tag{4.6}$$

Observe from (4.4) that $f(x) > -\infty$ and $f(y) > -\infty$. We can prove (4.6) by considering the following two cases.

In the case when $f(x) = \infty$ or $f(y) = \infty$, we have

$$f\left(ax + (1-a)\, y\right) \leq \infty = af\left(x\right) + (1-a)\, f\left(y\right),$$

which means (4.6) holds true.

In the second case, we assume that both $f(x)$ and $f(y)$ are finite. Then (4.3) implies that there exist $(\mu_1, x_1, \ldots, \mu_m, x_m)$ and $(\theta_1, y_1, \ldots, \theta_m, y_m)$ such that

$$\mu_i, \theta_i \in [0, 1],\ x_i, y_i \in \mathrm{dom}\, f_i \text{ for all } i = 1, \ldots, m$$

and

$$\sum_{i=1}^{m} \mu_i = 1, \qquad \sum_{i=1}^{m} \mu_i x_i = x,$$
$$\sum_{i=1}^{m} \theta_i = 1, \qquad \sum_{i=1}^{m} \theta_i y_i = y.$$

Fix any $i = 1, \ldots, m$. Define

$$\gamma_i := a\mu_i + (1-a)\,\theta_i \in [0,1].$$

Observe that $\gamma_i = 0$ if and only if $\mu_i = \theta_i = 0$. Moreover, let

$$z_i := \begin{cases} x_i & \text{if } \gamma_i = 0, \\ \frac{a\mu_i}{\gamma_i} x_i + \frac{(1-a)\theta_i}{\gamma_i} y_i & \text{if } \gamma_i \in (0,1], \end{cases}$$

where $\frac{a\mu_i}{\gamma_i}, \frac{(1-a)\theta_i}{\gamma_i} \in [0,1]$ and

$$\frac{a\mu_i}{\gamma_i} + \frac{(1-a)\,\theta_i}{\gamma_i} = \frac{a\mu_i + (1-a)\,\theta_i}{\gamma_i} = \frac{\gamma_i}{\gamma_i} = 1.$$

Notice that $z_i \in \operatorname{dom} f_i$ because $\operatorname{dom} f_i$ is convex. Moreover, by straightforward calculation, it follows that

$$\sum_{k=1}^{m} \gamma_k = 1, \qquad \sum_{k=1}^{m} \gamma_k z_k = ax + (1-a)\,y.$$

Thus $(\gamma_1, z_1, \ldots, \gamma_m, z_m)$ satisfies the constraints of (4.3), and therefore

$$f\,(ax + (1-a)\,y) \leq \sum_{k=1}^{m} \left[\gamma_k f_k(z_k) + g_k^1(\gamma_k) + g_k^2(\gamma_k z_k)\right]. \tag{4.7}$$

Observe from the convexity of g_i^1 that

$$g_i^1(\gamma_i) = g_i^1(a\mu_i + (1-a)\,\theta_i) \leq a g_i^1(\mu_i) + (1-a)\,g_i^1(\theta_i).$$

We consider the following two cases for γ_i. In the case when $\gamma_i = 0$, we have from $\mu_i = 0$ and $\theta_i = 0$ that

$$\gamma_i f_i(z_i) = 0 = a\mu_i f_i(x_i) + (1-a)\,\theta_i f_i(y_i),$$
$$g_i^2(\gamma_i z_i) = g_i^2(0) = a g_i^2(\mu_i x_i) + (1-a)\,g_i^2(\theta_i y_i).$$

In the case when $\gamma_i \in (0,1]$, the convexity of f_i and g_i^2 implies

$$\gamma_i f_i(z_i) = \gamma_i f_i\left(\frac{a\mu_i}{\gamma_i}x_i + \frac{(1-a)\,\theta_i}{\gamma_i}y_i\right) \le a\mu_i f_i(x_i) + (1-a)\,\theta_i f_i(y_i),$$
$$g_i^2(\gamma_i z_i) = g_i^2(a\mu_i x_i + (1-a)\,\theta_i y_i) \le a g_i^2(\mu_i x_i) + (1-a)\,g_i^2(\theta_i y_i).$$

Therefore, we can conclude that

$$\sum_{k=1}^{m}\Big[\gamma_k f_k(z_k) + g_k^1(\gamma_k) + g_k^2(\gamma_k z_k)\Big] \le a\sum_{k=1}^{m}\Big[\mu_k f_k(x_k) + g_k^1(\mu_k) + g_k^2(\mu_k x_k)\Big]$$
$$+ (1-a)\sum_{k=1}^{m}\Big[\theta_k f_k\,(y_k) + g_i^1(\theta_k) + g_k^2(\theta_k y_k)\Big].$$

Combining this with (4.7), we have

$$f\,(ax + (1-a)\,y) \le a\sum_{k=1}^{m}\Big[\mu_k f_k(x_k) + g_k^1(\mu_k) + g_k^2(\mu_k x_k)\Big]$$
$$+ (1-a)\sum_{k=1}^{m}\Big[\theta_k f_k\,(y_k) + g_i^1(\theta_k) + g_k^2(\theta_k y_k)\Big].$$

Taking infimum on both sides, it follows that

$$f\,(ax + (1-a)\,y) \le af\,(x) + (1-a)\,f\,(y)\,,$$

which completes the proof of (4.6). Thus f is convex.

Finally, we are going to prove (4.5). For any $x \in \mathbb{R}$, the value $f(x)$ in is finite if and only if there exists $(\lambda_1, x_1, \ldots, \lambda_m, x_m)$ that satisfies the constraints in (4.3). Moreover, the constraints in (4.3) are satisfied for some $(\lambda_1, x_1, \ldots, \lambda_m, x_m)$ if and only if $x \in \operatorname{co}(\cup_{i=1,\ldots,m}\operatorname{dom} f_i)$. Therefore (4.5) holds true. □

4.2 Existence of solution

This section is devoted to showing that the infimum in (4.3) is attained for all $x \in \operatorname{dom} f$, in other words, there exists a solution to the minimisation problem in (4.3) for every $x \in \operatorname{dom} f$. Moreover, we will also show that f is continuous on $\operatorname{dom} f$. Firstly, we will introduce an auxiliary set $E_f \subseteq \mathbb{R}^2$ in (4.8). After that, a number of technical results will be provided for establishing the closedness of E_f; see Theorem 4.11. Then we will show that $E_f = \operatorname{epi} f$ in Theorem 4.12. Finally, we will present the main result of this section in Theorem 4.13.

Define $E_f \subseteq \mathbb{R}^2$ as

$$E_f := \left\{ \left(\sum_{i=1}^m \lambda_i x_i^1, \sum_{i=1}^m \left(\lambda_i x_i^2 + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i^1) \right) \right) \middle| \right.$$
$$\left. \lambda_i \in [0,1], (x_i^1, x_i^2) \in \operatorname{epi} f_i \; \forall i = 1, \ldots, m, \sum_{i=1}^m \lambda_i = 1 \right\}. \quad (4.8)$$

The following result says that $(0,b) \in E_f^\infty$ for all $b \geq 0$, and this property will be used in the proof of Theorem 4.12.

Proposition 4.4. *We have* $\{(0,b) \in \mathbb{R}^2 \mid b \geq 0\} \subseteq E_f^\infty$.

Proof. Let $b \geq 0$, and fix any $\epsilon \geq 0$ and $(z^1, z^2) \in E_f$. Since $(z^1, z^2) \in E_f$, there exists $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ and $(x_1^1, x_1^2), \ldots, (x_m^1, x_m^2) \in \mathbb{R}^2$ such that

$$\sum_{i=1}^m \lambda_i = 1, \qquad \lambda_i \in [0,1], (x_i^1, x_i^2) \in \operatorname{epi} f_i \text{ for all } i = 1, \ldots, m$$

and

$$(z^1, z^2) = \left(\sum_{i=1}^m \lambda_i x_i^1, \sum_{i=1}^m \left(\lambda_i x_i^2 + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i^1) \right) \right).$$

Since $\lambda_1, \ldots, \lambda_m \in [0,1]$ and $\sum_{i=1}^m \lambda_i = 1$, there exists $i^* \in \{1, \ldots, m\}$ such that $\lambda_{i^*} > 0$. Define $(y_1^1, y_1^2), \ldots, (y_m^1, y_m^2)$ as

$$(y_i^1, y_i^2) = \begin{cases} (x_i^1, x_i^2 + \frac{\epsilon b}{\lambda_{i^*}}) & \text{if } i = i^* \\ (x_i^1, x_i^2) & \text{if } i \in \{1, \ldots, m\} \backslash \{i^*\}. \end{cases}$$

Notice that $(y_i^1, y_i^2) \in \operatorname{epi} f_i$ for all $i = 1, \ldots, m$, and

$$\sum_{i=1}^m \lambda_i y_i^1 = \sum_{i=1}^m \lambda_i x_i^1, \qquad \sum_{i=1}^m \lambda_i y_i^2 = \sum_{i=1}^m \lambda_i x_i^2 + \epsilon b.$$

By straightforward calculation, it follows that

$$(z^1, z^2) + \epsilon(0,b) = \left(\sum_{i=1}^m \lambda_i y_i^1, \sum_{i=1}^m \left(\lambda_i y_i^2 + g_i^1(\lambda_i) + g_i^2(\lambda_i y_i^1) \right) \right) \in E_f,$$

and the result follows. □

We are going to introduce a collection of sets $K_1, \ldots, K_m \subseteq \mathbb{R}^3$ which will be helpful for establishing the closedness of E_f in Theorem 4.11 below. For any $i = 1, \ldots, m$, we define $K_i \subseteq \mathbb{R}^3$ as

$$K_i := \left\{ \left(\lambda, \lambda x^1, \lambda x^2 + g_i^1(\lambda) + g_i^2(\lambda x^1) \right) \middle| \lambda \in [0,1], (x^1, x^2) \in \operatorname{epi} f_i \right\}. \quad (4.9)$$

Observe that $0 \in K_i$ because $g_i^1(0) = g_i^2(0) = 0$ by (4.2). Moreover, it follows directly from (4.9) that

$$(0, z^1, z^2) \in K_i \Longrightarrow z^1 = 0, z^2 = 0. \tag{4.10}$$

Then, for any $b > 0$, we have $(0, 0, b) \notin K_i^\infty$ because $0 + (0, 0, b) \notin K_i$. Lemmas 4.5-4.7 below will provide a number of properties of K_i.

The following technical result will be used in Propositions 4.8 and 4.10.

Lemma 4.5. *Let $i = 1, \ldots, m$, $b \geq 0$. If $(\lambda, z^1, z^2) \in K_i$ and $\lambda > 0$, then*

$$(\lambda, z^1, z^2) + (0, 0, b) \in K_i.$$

Proof. Suppose that $(\lambda, z^1, z^2) \in K_i$ and $\lambda > 0$. Observe from the definition of K_i in (4.9) that there exists $(x^1, x^2) \in \operatorname{epi} f_i$ such that

$$(\lambda, z^1, z^2) = \Big(\lambda, \lambda x^1, \lambda x^2 + g_i^1(\lambda) + g_i^2(\lambda x^1)\Big).$$

This implies that

$$(\lambda, z^1, z^2) + (0, 0, b) = \Big(\lambda, \lambda x^1, \lambda\left(x^2 + \tfrac{b}{\lambda}\right) + g_i^1(\lambda) + g_i^2(\lambda x^1)\Big) \in K_i$$

because $(x^1, x^2 + \frac{b}{\lambda}) \in \operatorname{epi} f_i$. This completes the proof. $\square$

The next result shows that $K_1, \ldots, K_m$ are convex sets, which implies that $\sum_{i=1}^m K_i$ is also convex (Rockafellar 1997, Theorem 3.1). Moreover, this result only relies on the convexity of the functions $f_1, g_1^1, g_1^2, \ldots, f_m, g_m^1, g_m^2$ and the condition (4.2).

Lemma 4.6. *For every $i = 1, \ldots, m$, the set K_i is convex.*

Proof. Fix any $i = 1, \ldots, m$, and fix any $a \in (0, 1)$ and $x', y' \in K_i$. From the definition of K_i in (4.9), there exist $\mu, \gamma \in [0, 1]$, $x = (x^1, x^2) \in \operatorname{epi} f_i$ and $y = (y^1, y^2) \in \operatorname{epi} f_i$ such that

$$\begin{aligned} x' &= \Big(\mu, \mu x^1, \mu x^2 + g_i^1(\mu) + g_i^2(\mu x^1)\Big), \\ y' &= \Big(\gamma, \gamma y^1, \gamma y^2 + g_i^1(\gamma) + g_i^2(\gamma y^1)\Big). \end{aligned}$$

Define

$$c := a\mu + (1 - a)\gamma \in [0, 1]. \tag{4.11}$$

Observe that $c = 0$ if and only if $\mu = \gamma = 0$. We are going to show that

$$a x' + (1 - a)\, y' \in K_i \tag{4.12}$$

by considering the following two cases for c. In the case when $c = 0$, we must have $\mu = \gamma = 0$. Combining this with $g_i^1(0) = g_i^2(0) = 0$ by (4.2), it follows that $x' = y' = 0$. This means

$$ax' + (1-a)y' = 0 \in K_i,$$

and hence (4.12) holds true. In the case when $c \in (0,1]$, let

$$\begin{aligned} z^1 &:= \frac{a\mu}{c}x^1 + \frac{(1-a)\gamma}{c}y^1, \\ \epsilon &:= a\left(g_i^1(\mu) + g_i^2(\mu x^1)\right) + (1-a)\left(g_i^1(\gamma) + g_i^2(\gamma y^1)\right) - g_i^1(c) - g_i^2(cz^1), \\ z^2 &:= \frac{a\mu}{c}x^2 + \frac{(1-a)\gamma}{c}y^2 + \frac{\epsilon}{c}. \end{aligned}$$

Notice that

$$cz^1 = a\mu x^1 + (1-a)\,\gamma y^1. \tag{4.13}$$

Moreover, it follows from the constructions of z^2 and ϵ above that

$$\begin{aligned} cz^2 + g_i^1(c) + g_i^2(cz^1) = a&\left(\mu x^2 + g_i^1(\mu) + g_i^2(\mu x^1)\right) \\ &+ (1-a)\left(\gamma y^2 + g_i^1(\gamma) + g_i^2(\gamma y^1)\right). \end{aligned} \tag{4.14}$$

Then the presentations of the values c, cz^1, and $cz^2 + g_i^1(c) + g_i^2(cz^1)$ in (4.11), (4.13), and (4.14) lead to

$$ax' + (1-a)\,y' = \left(c, cz^1, cz^2 + g_i^1(c) + g_i^2(cz^1)\right).$$

Then, in order to prove (4.12), it is enough to show that $(z^1, z^2) \in \operatorname{epi} f_i$. Observe from the definitions of z^1 and z^2 that

$$(z^1, z^2) = \frac{a\mu}{c}x + \frac{(1-a)\,\gamma}{c}y + \left(0, \frac{\epsilon}{c}\right),$$

where $\frac{a\mu}{c}, \frac{(1-a)\gamma}{c} \in [0,1]$ and

$$\frac{a\mu}{c} + \frac{(1-a)\gamma}{c} = \frac{a\mu + (1-a)\gamma}{c} = \frac{c}{c} = 1.$$

Then the convexity of $\operatorname{epi} f_i$ together with $x, y \in \operatorname{epi} f_i$ gives

$$\frac{a\mu}{c}x + \frac{(1-a)\,\gamma}{c}y \in \operatorname{epi} f_i.$$

By the definition of ϵ together with the presentations of c and cz^1 in (4.11) and (4.13), we have

$$\epsilon = a\left(g_i^1(\mu) + g_i^2(\mu x^1)\right) + (1-a)\left(g_i^1(\gamma) + g_i^2(\gamma y^1)\right) \\ - g_i^1(a\mu + (1-a)\gamma) - g_i^2(a\mu x^1 + (1-a)\,\gamma y^1).$$

Thus $\epsilon \geq 0$ because g_i^1 and g_i^2 are convex. This implies that $\frac{\epsilon}{c} \geq 0$ which means $(z^1, z^2) \in \operatorname{epi} f_i$. Therefore (4.12) holds true, and hence K_i is convex. $\square$

The next result gives an explicit expression for $\operatorname{cl} K_i$ for all $i = 1, \ldots, m$, where $\operatorname{cl} A$ is the closure of a given set A.

Lemma 4.7. *For all $i = 1, \ldots, m$, we have*

$$\operatorname{cl} K_i = K_i \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\}.$$

Proof. Fix any $i = 1, \ldots, m$. Define the following auxiliary set

$$K_i' := \left\{ (\lambda, \lambda x^1, \lambda x^2) \in \mathbb{R}^3 \middle| \, \lambda \geq 0, (x^1, x^2) \in \operatorname{epi} f_i \right\} \tag{4.15}$$

(cf. (4.9)). Observe from the definition of K_i' that

$$(0, z^1, z^2) \in K_i' \Longrightarrow (0, z^1, z^2) = 0.$$

Moreover, from Corollary 2.6.3 of Rockafellar (1997) and the comments following it, the family K_i' is the convex cone generated by

$$\left\{ (1, x^1, x^2) \middle| \, (x^1, x^2) \in \operatorname{epi} f_i \right\},$$

where $(\operatorname{epi} f_i)^\infty = \{ (0,b) |\, b \geq 0\}$ by (4.1). Then the closure of K_i' is given by

$$\operatorname{cl} K_i' = K_i' \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\} \tag{4.16}$$

(Rockafellar 1997, Theorem 8.2).

Firstly, we are going to show that

$$\operatorname{cl} K_i \subseteq K_i \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\}. \tag{4.17}$$

Fix any $(\lambda, z^1, z^2) \in \operatorname{cl} K_i$. Then there exists a sequence $(\lambda_k, z_k^1, z_k^2)_{k \in \mathbb{N}}$ in K_i converging to (λ, z^1, z^2). It follows from $\lambda_k \in [0,1]$ for all $k \in \mathbb{N}$ that

$$\lambda = \lim_{k \to \infty} \lambda_k \in [0,1]. \tag{4.18}$$

From the definition of K_i in (4.9), there exists a sequence $(x_k^1, x_k^2)_{k \in \mathbb{N}}$ in $\operatorname{epi} f_i$

such that

$$\left(\lambda_k, z_k^1, z_k^2\right) = \left(\lambda_k, \lambda_k x_k^1, \lambda_k x_k^2 + g_i^1(\lambda_k) + g_i^2(\lambda_k x_k^1)\right)$$

for all $k \in \mathbb{N}$. The continuity of g_i^1 and g_i^2 gives $g_i^1(\lambda) = \lim_{k\to\infty} g_i^1(\lambda_k)$ and $g_i^2(z^1) = \lim_{k\to\infty} g_i^2(z_k^1)$. Combining this with $z^2 = \lim_{k\to\infty} z_k^2$, it follows that

$$\begin{aligned} z^2 - g_i^1(\lambda) - g_i^2(z^1) &= \lim_{k\to\infty} \left(z_k^2 - g_i^1(\lambda_k) - g_i^2(z_k^1)\right) \\ &= \lim_{k\to\infty} \left(z_k^2 - g_i^1(\lambda_k) - g_i^2(\lambda_k x_k^1)\right) \\ &= \lim_{k\to\infty} \lambda_k x_k^2. \end{aligned}$$

Then $(\lambda_k, \lambda_k x_k^1, \lambda_k x_k^2)_{k\in\mathbb{N}}$ converges to $(\lambda, z^1, z^2 - g_i^1(\lambda) - g_i^2(z^1))$. Observe that $(\lambda_k, \lambda_k x_k^1, \lambda_k x_k^2)_{k\in\mathbb{N}}$ is a sequence in K_i' because $(x_k^1, x_k^2)_{k\in\mathbb{N}}$ is a sequence in $\operatorname{epi} f_i$. Thus

$$\left(\lambda, z^1, z^2 - g_i^1(\lambda) - g_i^2(z^1)\right) \in \operatorname{cl} K_i' = K_i' \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| b \geq 0 \right\} \tag{4.19}$$

by (4.16). We are going to show that

$$(\lambda, z^1, z^2) \in K_i \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| b \geq 0 \right\} \tag{4.20}$$

by considering the following two cases.

1. In the case when

   $$(\lambda, z^1, z^2 - g_i^1(\lambda) - g_i^2(z^1)) \in \left\{ (0,0,b) \in \mathbb{R}^3 \middle| b \geq 0 \right\},$$

   we must have $\lambda = z^1 = 0$, which implies

   $$-g_i^1(\lambda) - g_i^2(z^1) = -g_i^1(0) - g_i^2(0) = 0$$

   by (4.2). Thus

   $$\left(\lambda, z^1, z^2\right) = \left(\lambda, z^1, z^2 - g_i^1(\lambda) - g_i^2(z^1)\right) \in \left\{ (0,0,b) \in \mathbb{R}^3 \middle| b \geq 0 \right\},$$

   and hence (4.20) holds true.

2. In the case when $(\lambda, z^1, z^2 - g_i^1(\lambda) - g_i^2(z^1)) \in K_i'$, the definition of K_i' in (4.15) implies that there exists $(x^1, x^2) \in \operatorname{epi} f_i$ such that

   $$\left(\lambda, z^1, z^2 - g_i^1(\lambda) - g_i^2(z^1)\right) = \left(\lambda, \lambda x^1, \lambda x^2\right),$$

in other words,

$$\left(\lambda, z^1, z^2\right) = \left(\lambda, \lambda x^1, \lambda x^2 + g_i^1\left(\lambda\right) + g_i^2(\lambda x^1)\right).$$

Combining this with $\lambda \in [0,1]$ by (4.18), we have $(\lambda, z^1, z^2) \in K_i$ and hence (4.20) holds true.

This completes the proof of (4.17). It remains to show the opposite inclusion of (4.17). Fix any

$$(\lambda, z^1, z^2) \in K_i \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| b \geq 0 \right\}.$$

Clearly, when $(\lambda, z^1, z^2) \in K_i$, the vector $(\lambda, z^1, z^2) \in \operatorname{cl} K_i$. In the case when $(\lambda, z^1, z^2) = (0, 0, b)$ for some $b \geq 0$, we have from (4.16) that

$$(\lambda, z^1, z^2) \in \operatorname{cl} K_i'.$$

This implies that there exists a sequence $(\lambda_k, z_k^1, z_k^2)_{k \in \mathbb{N}}$ in K_i' converging to (λ, z^1, z^2). The definition of K_i' in (4.15) implies that, for any $k \in \mathbb{N}$, there exists $(x_k^1, x_k^2) \in \operatorname{epi} f_i$ such that

$$z_k^1 = \lambda_k x_k^1, \qquad\qquad z_k^2 = \lambda_k x_k^2.$$

Moreover, since $(\lambda_k)_{k \in \mathbb{N}}$ is a sequence in $[0, \infty)$ and $\lim_{k \to \infty} \lambda_k = \lambda = 0$, there exists $k^* \in \mathbb{N}$ such that $\lambda_k \in [0,1]$ for all $k \geq k^*$. Thus

$$(y_k)_{k \in \mathbb{N}} := \left(\lambda_{k^*+k}, z_{k^*+k}^1, z_{k^*+k}^2 + g_i^1(\lambda_{k^*+k}) + g_i^2(z_{k^*+k}^1)\right)_{k \in \mathbb{N}}$$

is a sequence in K_i. The continuity of g_i^1 and g_i^2 gives

$$\lim_{k \to \infty} g_i^1(\lambda_{k^*+k}) = g_i^1(\lambda) = g_i^1(0) = 0,$$
$$\lim_{k \to \infty} g_i^2(z_{k^*+k}^1) = g_i^2(z^1) = g_i^2(0) = 0.$$

Then $(y_k)_{k \in \mathbb{N}}$ converges to (λ, z^1, z^2). Therefore $(\lambda, z^1, z^2) \in \operatorname{cl} K_i$ which means that the opposite inclusion of (4.17) holds true. □

In Propositions 4.8-4.10 below, we are going to provide a number of properties of operations among $K_1, \ldots, K_m$. Moreover, these properties will be used in Theorem 4.11 which establishes the closedness of E_f defined in (4.8).

The proposition below shows that the $\operatorname{cl} \sum_{i=1}^m K_i$ can be written as the sum of $\operatorname{cl} K_1, \ldots, \operatorname{cl} K_m$.

Proposition 4.8. *We have*

$$\operatorname{cl}\sum_{i=1}^{m} K_i = \sum_{i=1}^{m} \operatorname{cl} K_i.$$

Proof. The main objective is to show that

$$(\operatorname{cl} K_i)^{\infty} = \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\} \text{ for all } i = 1, \ldots, m. \tag{4.21}$$

Taking (4.21) as given, we can prove the result as follows. Suppose that there exist $z_1, \ldots, z_m \in \mathbb{R}^3$ such that $\sum_{i=1}^{m} z_i = 0$ and

$$z_i \in (\operatorname{cl} K_i)^{\infty} \text{ for all } i = 1, \ldots, m.$$

Then $z_1 = \cdots = z_m = 0$, and this implies that

$$z_i \in \{0\} = (-(\operatorname{cl} K_i)^{\infty}) \cap (\operatorname{cl} K_i)^{\infty} \text{ for all } i = 1, \ldots, m,$$

where $K_1, \ldots, K_m$ are convex by Lemma 4.6. The result follows from Corollary 9.1.1 of Rockafellar (1997) and the comments following Corollary 8.4.1 of Rockafellar (1997).

Now, we are going to prove that (4.21) holds true. Fix any $i = 1, \ldots, m$. Firstly, we shall prove that

$$(\operatorname{cl} K_i)^{\infty} \subseteq \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\}. \tag{4.22}$$

Fix any $z_i = (z_i^1, z_i^2, z_i^3) \in (\operatorname{cl} K_i)^{\infty}$. Since $(\operatorname{cl} K_i)^{\infty}$ is the recession cone of $\operatorname{cl} K_i$ and $0 \in \operatorname{cl} K_i$, we have for all $\delta \geq 0$ that

$$\delta z_i = 0 + \delta z_i \in \operatorname{cl} K_i = K_i \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\} \tag{4.23}$$

(Lemma 4.7). Combining the definition of K_i in (4.9) and the fact that (4.23) holds true for all $\delta \geq 0$, the first component of z_i must be $z_i^1 = 0$. Then (4.10) implies that $z_i = 0$ as long as $z_i \in K_i$. Thus

$$z_i \in \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\},$$

which means that (4.22) holds true.

It remains to prove the opposite inclusion of (4.22). Fix any $b \geq 0$, $\delta \geq 0$ and $z_i = (z_i^1, z_i^2, z_i^3) \in \operatorname{cl} K_i$. Then Lemma 4.7 gives

$$z_i \in K_i \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\} = (K_i \backslash \{0\}) \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\}.$$

Consider the following two cases. If $z_i \in K_i \backslash \{0\}$, then $z_i^1 > 0$ and

$$z_i + \delta(0,0,b) = z_i + (0,0,\delta b) \in K_i$$

(Lemma 4.5). If $z_i \in \{(0,0,b) \in \mathbb{R}^3 \mid b \geq 0\}$, then

$$z_i + \delta(0,0,b) \in \{\,(0,0,b)|\, b \geq 0\}\,.$$

Thus, we always have

$$z_i + \delta(0,0,b) \in K_i \cup \{\,(0,0,b)|\, b \geq 0\} = \operatorname{cl} K_i$$

(Lemma 4.7). This means $(0,0,b) \in (\operatorname{cl} K_i)^\infty$, and hence the opposite inclusion of (4.22) holds true. Therefore (4.21) holds true, and the result follows. ☐

Define the hyperplane

$$M := \left\{ \left(1, x^1, x^2\right) \middle| \, (x^1, x^2) \in \mathbb{R}^2 \right\} \tag{4.24}$$

in $\mathbb{R}^3$. Observe from (4.9) and (4.8) that

$$\begin{aligned}\left(\sum_{i=1}^m K_i\right) \bigcap M = &\left\{ \left(1, \sum_{i=1}^m \lambda_i x_i^1, \sum_{i=1}^m \left(\lambda_i x_i^2 + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i^1)\right)\right) \right| \\ &\qquad \left. \lambda_i \in [0,1], (x_i^1, x_i^2) \in \operatorname{epi} f_i \, \forall i = 1, \ldots, m, \sum_{i=1}^m \lambda_i = 1 \right\} \\ = &\{1\} \times E_f. \end{aligned} \tag{4.25}$$

The result below will be used in Theorem 4.11 which shows that E_f is closed.

Proposition 4.9. *We have*

$$\left(\operatorname{cl} \sum_{i=1}^m K_i\right) \bigcap M = \operatorname{cl}\left(\left(\sum_{i=1}^m K_i\right) \bigcap M\right).$$

Proof. Observe from Lemma 4.6 that $\sum_{i=1}^m K_i$ is convex. The objective is to show that M contains an element of $\operatorname{ri}(\sum_{i=1}^m K_i)$, where $\operatorname{ri}(A)$ is the relative interior of a given set A. Then the result follows from Corollary 6.5.1 of Rockafellar (1997).

It follows from $0 \in K_i$ for all $i = 1, \ldots, m$ that $0 \in \sum_{i=1}^m K_i$ which means $\sum_{i=1}^m K_i \neq \emptyset$. Then Theorem 6.2 of Rockafellar (1997) gives

$$\operatorname{ri}\left(\sum_{i=1}^m K_i\right) \neq \emptyset.$$

Fix any $z = (z^1, z^2, z^3) \in \operatorname{ri}(\sum_{i=1}^m K_i)$. Then $z \in \sum_{i=1}^m K_i$ which means that there exist $\lambda_1, \ldots, \lambda_m \in [0,1]$ and $(x_1^1, x_1^2) \in \operatorname{epi} f_1, \ldots, (x_m^1, x_m^2) \in \operatorname{epi} f_m$ such that

$$z = \left(z^1, z^2, z^3\right) = \left(\sum_{i=1}^m \lambda_i, \sum_{i=1}^m \lambda_i x_i^1, \sum_{i=1}^m \left(\lambda_i x_i^2 + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i^1)\right)\right).$$

Observe that $z^1 \geq 0$, and we are going to consider the following two cases based on the value of z^1. In the case when $z^1 \geq 1$, we have $\frac{z}{z^1} \in M$ and

$$\frac{z}{z^1} = \frac{1}{z^1} z + \left(1 - \frac{1}{z^1}\right) \times 0 \in \operatorname{ri}\left(\sum_{i=1}^m K_i\right)$$

(Rockafellar 1997, Theorem 6.1). In the case when $z^1 \in [0,1)$, let

$$\mu := \frac{1}{m}(2 - z^1) \in (0,1]$$

and

$$\begin{aligned} z' = (z'^1, z'^2, z'^3) &:= \left(\sum_{i=1}^m \mu, \sum_{i=1}^m \mu x_i^1, \sum_{i=1}^m \left(\mu x_i^2 + g_i^1(\mu) + g_i^2(\mu x_i^1)\right)\right) \\ &= \left(2 - z^1, \mu \sum_{i=1}^m x_i^1, \sum_{i=1}^m \left(\mu x_i^2 + g_i^1(\mu) + g_i^2(\mu x_i^1)\right)\right). \end{aligned}$$

Observe from the definition of z' that

$$z' \in \sum_{i=1}^m K_i \subseteq \operatorname{cl}\left(\sum_{i=1}^m K_i\right).$$

Define

$$z^* = (z^{*1}, z^{*2}, z^{*3}) := \tfrac{1}{2} z + \tfrac{1}{2} z' \in \operatorname{ri}\left(\sum_{i=1}^m K_i\right)$$

(Rockafellar 1997, Theorem 6.1). Observe that

$$z^{*1} = \tfrac{1}{2} z^1 + \tfrac{1}{2} z'^1 = \tfrac{1}{2} z^1 + \tfrac{1}{2}(2 - z^1) = 1$$

which means $z^* \in M$. Thus M contains an element of $\operatorname{ri}(\sum_{i=1}^m K_i)$, and the result follows. □

The next result connects $\sum_{i=1}^m (\operatorname{cl} K_i)$ and E_f.

Proposition 4.10. *We have*

$$\left(\sum_{i=1}^m \operatorname{cl} K_i\right) \bigcap M = \{1\} \times E_f.$$

Proof. Observe from (4.25) that

$$\{1\} \times E_f \subseteq \left(\sum_{i=1}^{m} \operatorname{cl} K_i \right) \bigcap M. \tag{4.26}$$

It is therefore sufficient to show that the opposite inclusion of (4.26) holds true. Fix any $z = (z^1, z^2, z^3) \in (\sum_{i=1}^{m} \operatorname{cl} K_i) \cap M$. Then Lemma 4.7 together with the definition of M in (4.24) implies that there exist

$$z_1 = (z_1^1, z_1^2, z_1^3), \dots, z_m = (z_m^1, z_m^2, z_m^3) \in \mathbb{R}^3$$

such that

$$\sum_{i=1}^{m} z_i = z, \quad \sum_{i=1}^{m} z_i^1 = 1, \quad z_i \in K_i \cup \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\} \ \forall i = 1, \dots, m.$$

Since $\sum_{i=1}^{m} z_i^1 = 1$ and $z_i^1 \geq 0$ for all $i = 1, \dots, m$, there exists $i^* \in \{1, \dots, m\}$ such that $z_{i^*}^1 > 0$. Define two subsets of $\{1, \dots, m\}$ as

$$A_0 := \left\{ i \in \{1, \dots, m\} \middle| z_i^1 = 0 \right\},$$
$$A_1 := \{1, \dots, m\} \backslash (A_0 \cup \{i^*\}).$$

Notice that $\{i^*\}, A_0, A_1$ are pairwise disjoint and $\{i^*\} \cup A_0 \cup A_1 = \{1, \dots, m\}$. Moreover, for any $i \in A_1 \cup \{i^*\}$, we have $z_i^1 > 0$ which means $z_i \in K_i$. In addition, it follows from (4.10) that $z_i \in \{(0,0,b) \in \mathbb{R}^3 \mid b \geq 0\}$ for all $i \in A_0$, which implies

$$\textstyle\sum_{k \in A_0} z_k \in \left\{ (0,0,b) \in \mathbb{R}^3 \middle| \, b \geq 0 \right\}.$$

Combining this with Lemma 4.5, it follows that $z_{i^*} + \sum_{k \in A_0} z_k \in K_{i^*}$. Now, we define $y_1, \dots, y_m \in \mathbb{R}^3$ as

$$y_i := \begin{cases} z_{i^*} + \sum_{k \in A_0} z_k & \text{if } i = i^*, \\ 0 & \text{if } i \in A_0, \\ z_i & \text{if } i \in A_1. \end{cases}$$

Observe that

$$z = \sum_{i=1}^{m} z_i = z_{i^*} + \sum_{k \in A_0} z_k + \sum_{k \in A_1} z_k = \sum_{i=1}^{m} y_i,$$

where $y_i \in K_i$ for all $i = 1, \dots, m$. Combining this with $z \in M$, we have

$$z \in \left(\sum_{i=1}^{m} K_i \right) \bigcap M = \{1\} \times E_f$$

by (4.25). Thus, the opposite inclusion of (4.26) holds true. □

The result below establishes the closedness of E_f.

Theorem 4.11. *The set E_f is closed.*

Proof. Observe that

$$\{1\} \times \operatorname{cl} E_f = \operatorname{cl}(\{1\} \times E_f) = \operatorname{cl} \left(\left(\sum_{i=1}^{m} K_i \right) \bigcap M \right)$$

by (4.25). Then Proposition 4.9 gives

$$\{1\} \times \operatorname{cl} E_f = \left(\operatorname{cl} \sum_{i=1}^{m} K_i \right) \bigcap M = \left(\sum_{i=1}^{m} \operatorname{cl} K_i \right) \bigcap M$$

(Proposition 4.8). Combining this with Proposition 4.10, it follows that

$$\{1\} \times \operatorname{cl} E_f = \{1\} \times E_f.$$

Thus $\operatorname{cl} E_f = E_f$, in other words, the set E_f is closed. □

The result below shows that E_f is the epigraph of f.

Theorem 4.12. *We have $E_f = \operatorname{epi} f$.*

Proof. Firstly, we shall prove that

$$E_f \subseteq \operatorname{epi} f. \tag{4.27}$$

Suppose that $(x, y) \in E_f$. Then the definition of E_f in (4.8) implies that there exist $\lambda_i \in [0, 1]$ and $x_i = (x_i^1, x_i^2) \in \operatorname{epi} f_i$ for every $i = 1, \dots, m$ such that

$$\sum_{i=1}^{m} \lambda_i = 1, \quad \sum_{i=1}^{m} \lambda_i x_i^1 = x, \quad \sum_{i=1}^{m} \left(\lambda_i x_i^2 + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i^1) \right) = y.$$

For any $i = 1, \dots, m$, it follows from $(x_i^1, x_i^2) \in \operatorname{epi} f_i$ that $x_i^2 \geq f_i(x_i^1)$. Combining this with the definition of $f(x)$ in (4.3), it follows that

$$y \geq \sum_{i=1}^{m} \left(\lambda_i f_i(x_i^1) + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i^1) \right) \geq f(x),$$

which implies $(x, y) \in \operatorname{epi} f$. Thus (4.27) holds true.

It remains to show that the opposite inclusion of (4.27) holds true. Suppose that $(x, y) \in \operatorname{epi} f$. Then $f(x) \leq y < \infty$. By (4.4), the value $f(x)$ is finite. Moreover, the definition of $f(x)$ in (4.3) implies that there exists a sequence $(\lambda_1^k, x_1^k, \ldots, \lambda_m^k, x_m^k)_{k \in \mathbb{N}}$ in $\mathbb{R}^{2m}$ such that

$$\lambda_1^k, \ldots, \lambda_m^k \in [0,1], x_1^k \in \operatorname{dom} f_1, \ldots, x_m^k \in \operatorname{dom} f_m \text{ for all } k \in \mathbb{N},$$
$$\sum_{i=1}^m \lambda_i^k = 1, \sum_{i=1}^m \lambda_i^k x_i^k = x \text{ for all } k \in \mathbb{N}$$

and $\lim_{k \to \infty} y^k = f(x)$ where

$$y^k := \sum_{i=1}^m \left(\lambda_i^k f_i(x_i^k) + g_i^1(\lambda_i^k) + g_i^2(\lambda_i^k x_i^k) \right) \in \mathbb{R} \text{ for all } k \in \mathbb{N}. \tag{4.28}$$

For any $k \in \mathbb{N}$, we have $(x_i^k, f_i(x_i^k)) \in \operatorname{epi} f_i$ for all $i = 1, \ldots, m$, which implies

$$\left(\sum_{i=1}^m \lambda_i^k x_i^k, y^k \right) \in E_f$$

Combining this with

$$\lim_{k \to \infty} \left(\sum_{i=1}^m \lambda_i^k x_i^k, y^k \right) = \lim_{k \to \infty} \left(x, y^k \right) = (x, f(x)),$$

it follows that $(x, f(x)) \in \operatorname{cl} E_f$. Then $(x, f(x)) \in E_f$ because E_f is closed (Theorem 4.11). Thus

$$(x, y) = (x, f(x) + (y - f(x))) = (x, f(x)) + (0, y - f(x)) \in E_f$$

by $y - f(x) \geq 0$ and Proposition 4.4. Therefore, the opposite inclusion of (4.27) holds true. $\square$

Finally, the following result shows that the infimum in (4.3) is attained for every $x \in \operatorname{dom} f$ (i.e. there exists a solution to the minimisation problem in (4.3) for all $x \in \operatorname{dom} f$). This result also establishes the continuity of f.

Theorem 4.13. *The infimum in* (4.3) *is attained for all* $x \in \operatorname{dom} f$. *In addition, the function* f *is continuous on* $\operatorname{dom} f$.

Proof. Firstly, we shall prove that the infimum in (4.3) is attained for every $x \in \operatorname{dom} f$. Let $x \in \operatorname{dom} f$. Then $f(x)$ is finite, which means

$$(x, f(x)) \in \operatorname{epi} f = E_f$$

(Theorem 4.12). Then it follows from the definition of E_f in (4.8) that there exists $\lambda_1, \ldots, \lambda_m \in [0,1]$ and $(x_1, y_1) \in \operatorname{epi} f_1, \ldots, (x_m, y_m) \in \operatorname{epi} f_m$ such that $\sum_{i=1}^m \lambda_i = 1$ and

$$(x, f(x)) = \left(\sum_{i=1}^m \lambda_i x_i, \sum_{i=1}^m \Big(\lambda_i y_i + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i) \Big) \right).$$

Notice that $(\lambda_1, x_1, \ldots, \lambda_m, x_m)$ satisfies the constraints of the problem (4.3). For every $i = 1, \ldots, m$, we have $y_i \geq f_i(x_i)$ because $(x_i, y_i) \in \operatorname{epi} f_i$. Then

$$f(x) \geq \sum_{i=1}^m \Big(\lambda_i f_i(x_i) + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i) \Big).$$

Moreover, the definition of $f(x)$ in (4.3) gives

$$f(x) \leq \sum_{i=1}^m \Big(\lambda_i f_i(x_i) + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i) \Big).$$

Therefore

$$f(x) = \sum_{i=1}^m (\lambda_i f_i(x_i) + g_i^1(\lambda_i) + g_i^2(\lambda_i x_i)),$$

and hence the infimum in (4.3) is attained.

By Theorems 4.3, 4.11 and 4.12, the function f is proper and convex, and $\operatorname{epi} f$ is closed (i.e. f is lower semicontinuous). Then f must be continuous on $\operatorname{dom} f$ (Lemma A.2). □

4.3 An example with an entropy function

In this section, we will consider two special examples of the minimisation problem in (4.3) for $m = 2$. Throughout this section, the functions g_1^1 and g_2^1 are defined as

$$g_1^1(y) = \begin{cases} y \ln \frac{y}{p_1} & \text{if } y \geq 0, \\ \infty & \text{otherwise,} \end{cases} \qquad g_2^1(y) = \begin{cases} y \ln \frac{y}{p_2} & \text{if } y \geq 0, \\ \infty & \text{otherwise,} \end{cases} \tag{4.29}$$

where $p_1, p_2 > 0$ are a given parameters. Moreover, the functions g_1^2 and g_2^2 are set to be $g_1^2 = g_2^2 = 0$. In Section 4.3.1, we will consider the situation when functions f_1 and f_2 in (4.3) are affine on their effective domains. By considering all different cases of the given parameters, we will explicitly present the solutions to (4.3) with $x \in \operatorname{dom} f$, so that these solutions can be easily calculated by a programming tool. In Section 4.3.2, the functions f_1 and f_2 are allowed to be piecewise linear on their effective domains. By using the result established in Section 4.3.1, we will provide a method to construct a solution to (4.3) with $x \in \operatorname{dom} f$.

4.3.1 Affine case

In this section, the functions f_1 and f_2 in (4.3) are defined as follows. For each $i = 1, 2$, let $\alpha_i, \beta_i \in \mathbb{R}$, and let $[b_i, a_i] \subseteq \mathbb{R}$ for some $b_i \leq a_i$. Moreover, let

$$f_i(y) = \begin{cases} \alpha_i y + \beta_i & \text{if } y \in [b_i, a_i], \\ \infty & \text{if } y \in \mathbb{R} \backslash [b_i, a_i]. \end{cases} \tag{4.30}$$

Then f_i is an affine function with slope α_i on $\operatorname{dom} f_i = [b_i, a_i]$. For any $x \in \mathbb{R}$, it follows from (4.3) and $g_1^2 = g_2^2 = 0$ that

$$f(x) = \inf \left\{ \sum_{i=1,2} \Big(\lambda_i f_i(x_i) + g_i^1(\lambda_i) \Big) \middle| \lambda_i \in [0,1], x_i \in [b_i, a_i] \, \forall i = 1, 2, \sum_{i=1,2} \lambda_i = 1, \sum_{i=1,2} \lambda_i x_i = x \right\}. \tag{4.31}$$

From (4.5), the effective domain of f can be written as

$$\operatorname{dom} f = \operatorname{co} \left([b_1, a_1] \cup [b_2, a_2] \right) = \left[b_1 \wedge b_2, a_1 \vee a_2 \right],$$

where $c_1 \wedge c_2 = \min\{c_1, c_2\}$ and $c_1 \vee c_2 = \max\{c_1, c_2\}$ for any $c_1, c_2 \in \mathbb{R}$. For the remainder of this section, we assume that $x \in \operatorname{dom} f$ which means

$$x \in \left[b_1 \wedge b_2, a_1 \vee a_2 \right]. \tag{4.32}$$

Then $f(x)$ must be finite. Moreover, from Theorem 4.13, there exists a solution to the minimisation problem in (4.31) with parameter x. In the remainder of this section, our objective is to construct a solution to this problem.

In (4.31), taking into account the constraints, the value

$$\sum_{i=1,2} \Big(\lambda_i f_i(x_i) + g_i^1(\lambda_i) \Big)$$

is determined by the control variables λ_1 and x_1. This means that we can reduce the dimensionality of (4.31), from four to two. In (4.37) below, we will provide a feasible set of (λ_1, x_1) in the problem (4.31). Firstly, we define

$$\bar{\mathcal{Q}}_x := \left\{ \lambda_1 \in [0,1] \, | \exists x_1 \in [b_1, a_1], x_2 \in [b_2, a_2] : \lambda_1 x_1 + (1 - \lambda_1) x_2 = x \right\} \tag{4.33}$$

as the feasible set of the control variable λ_1 in (4.31). Observe from $x \in \operatorname{dom} f$ that $\bar{\mathcal{Q}}_x \neq \emptyset$. Moreover, we have

$$1 \in \bar{\mathcal{Q}}_x \Longleftrightarrow x \in [b_1, a_1], \tag{4.34}$$

$$0 \in \bar{\mathcal{Q}}_x \Longleftrightarrow x \in [b_2, a_2]. \tag{4.35}$$

It turns out that $\bar{\mathcal{Q}}_x$ is a closed subinterval of $[0,1]$; see Lemma 4.14 below. For any $\lambda_1 \in \bar{\mathcal{Q}}_x$, we define

$$\mathcal{Z}_{\lambda_1,x} := \{x_1 \,| x_1 \in [b_1, a_1]\,, x_2 \in [b_2, a_2]\,, \lambda_1 x_1 + (1-\lambda_1)\, x_2 = x\} \tag{4.36}$$

as the collection of x_1 that satisfies the constraints of (4.31) with λ_1 fixed. We have $\mathcal{Z}_{\lambda_1,x} \neq \emptyset$ as long as $\lambda_1 \in \bar{\mathcal{Q}}_x$. If $1 \in \bar{\mathcal{Q}}_x$, then $\mathcal{Z}_{1,x} = \{x\}$. Moreover, if $0 \in \bar{\mathcal{Q}}_x$, then $\mathcal{Z}_{0,x} = [b_1, a_1]$. Observe that

$$\left\{(\lambda_1, x_1) \,\middle|\, \lambda_1 \in \bar{\mathcal{Q}}_x, x_1 \in \mathcal{Z}_{\lambda_1,x}\right\} \tag{4.37}$$

is the set of all possible (λ_1, x_1) that satisfies the constraits of (4.31). In Lemma 4.14 below, we provide a method to compute $\bar{\mathcal{Q}}_x$ explicitly. By considering three possible cases of the relationship between x and $[b_2, a_2]$, the quantity $q_x^{\min} \in [0,1]$ is defined as

$$q_x^{\min} := \begin{cases} \frac{x-b_2}{b_1-b_2} & \text{if } b_1 \leq x < b_2, \\ 0 & \text{if } b_2 \leq x \leq a_2, \\ \frac{x-a_2}{a_1-a_2} & \text{if } a_2 < x \leq a_1. \end{cases} \tag{4.38}$$

Similarly, by considering three possible situations of the relationship between x and $[b_1, a_1]$, we define $q_x^{\max} \in [0,1]$ as

$$q_x^{\max} := \begin{cases} \frac{x-b_2}{b_1-b_2} & \text{if } b_2 \leq x < b_1, \\ 1 & \text{if } b_1 \leq x \leq a_1, \\ \frac{x-a_2}{a_1-a_2} & \text{if } a_1 < x \leq a_2. \end{cases} \tag{4.39}$$

Lemma 4.14. *We have* $0 \leq q_x^{\min} \leq q_x^{\max} \leq 1$. *Moreover, the family* $\bar{\mathcal{Q}}_x$ *is given by*

$$\bar{\mathcal{Q}}_x = \left[q_x^{\min}, q_x^{\max}\right],$$

which means that $\bar{\mathcal{Q}}_x$ *is a closed subinterval of* $[0,1]$.

The proof of Lemma 4.14 above will be provided at the end of this section.

Now, we define a subset of $\bar{\mathcal{Q}}_x$ as

$$\mathcal{Q}_x := \bar{\mathcal{Q}}_x \cap (0,1) = \left[q_x^{\min}, q_x^{\max}\right] \cap (0,1). \tag{4.40}$$

which is the collection of $\lambda_1 \in (0,1)$ that satisfies the constraints of (4.31). Observe that $\mathcal{Q}_x = \bar{\mathcal{Q}}_x \backslash \{0,1\}$, and hence $\mathcal{Q}_x = \bar{\mathcal{Q}}_x$ if and only if $0 \notin \bar{\mathcal{Q}}_x$ and $1 \notin \bar{\mathcal{Q}}_x$. It is possible that $\mathcal{Q}_x = \emptyset$. For example, in the case when $a_2 < b_1 \leq x = a_1$, the definitions of $q_x^{\min}$ and $q_x^{\max}$ in (4.38)-(4.39) gives $q_x^{\min} = q_x^{\max} = 1$ which implies $\mathcal{Q}_x = \emptyset$. The family $\mathcal{Q}_x$ is a subinterval of $(0,1)$ as long as $\mathcal{Q}_x$ is not empty. For any $\gamma \in (0,1)$ and $z \in \mathbb{R}$, we define

$$\psi_{\gamma,x}(z) := \frac{x - \gamma z}{1-\gamma}, \tag{4.41}$$

$$\psi_{\gamma,x}^{-1}(z) := \frac{x-(1-\gamma)z}{\gamma}, \tag{4.42}$$

where $z \mapsto \psi_{\gamma,x}^{-1}(z)$ is the inverse function of $z \mapsto \psi_{\gamma,x}(z)$. As long as $\mathcal{Q}_x \neq \emptyset$, we can compute $\mathcal{Z}_{\gamma,x}$ for any $\gamma \in \mathcal{Q}_x$ by using the formula provided in the following result.

Proposition 4.15. *If $\gamma \in \mathcal{Q}_x$, then $\mathcal{Z}_{\gamma,x} \neq \emptyset$ and*

$$\mathcal{Z}_{\gamma,x} = \{z \,|z \in [b_1,a_1], \psi_{\gamma,x}(z) \in [b_2,a_2]\} = \left[b_1 \vee \psi_{\gamma,x}^{-1}(a_2), a_1 \wedge \psi_{\gamma,x}^{-1}(b_2)\right]$$

which is a closed subinterval of $[b_1,a_1]$.

Proof. Suppose that $\gamma \in \mathcal{Q}_x$. Then $\gamma \in \bar{\mathcal{Q}}_x$. It follows from the comments following (4.36) that $\mathcal{Z}_{\gamma,x} \neq \emptyset$. Moreover, by (4.36), the family $\mathcal{Z}_{\gamma,x}$ can be written as

$$\mathcal{Z}_{\gamma,x} = \{z \,|z \in [b_1,a_1], x_2 \in [b_2,a_2], \gamma z + (1-\gamma)\, x_2 = x\}.$$

Since $\gamma \in \mathcal{Q}_x \subseteq (0,1)$, it follows that

$$\begin{aligned}\mathcal{Z}_{\gamma,x} &= \left\{z \,\middle|\, z \in [b_1,a_1], x_2 \in [b_2,a_2], x_2 = \tfrac{x-\gamma z}{1-\gamma} = \psi_{\gamma,x}(z)\right\} \\ &= \{z \,|z \in [b_1,a_1], \psi_{\gamma,x}(z) \in [b_2,a_2]\}.\end{aligned}$$

Observe that

$$\begin{aligned}\psi_{\gamma,x}(z) \in [b_2,a_2] &\iff \tfrac{x-\gamma z}{1-\gamma} \in [b_2,a_2] \\ &\iff -\gamma z \in [(1-\gamma)b_2 - x, (1-\gamma)a_2 - x] \\ &\iff z \in \left[\tfrac{x-(1-\gamma)a_2}{\gamma}, \tfrac{x-(1-\gamma)b_2}{\gamma}\right] = \left[\psi_{\gamma,x}^{-1}(a_2), \psi_{\gamma,x}^{-1}(b_2)\right].\end{aligned}$$

Thus

$$\mathcal{Z}_{\gamma,x} = [b_1, a_1] \cap \left[\psi_{\gamma,x}^{-1}(a_2), \psi_{\gamma,x}^{-1}(b_2)\right] = \left[b_1 \vee \psi_{\gamma,x}^{-1}(a_2), a_1 \wedge \psi_{\gamma,x}^{-1}(b_2)\right],$$

which completes the proof. $\square$

We shall first reduce the dimensionality of the problem (4.31) and then work on the simplified problem. In (4.37), we provided the feasible set of (λ_1, x_1) in the problem (4.31). Taking into account the constraints of (4.31), we are going to express the value

$$\sum_{i=1,2} (\lambda_i f_i(x_i) + g_i^1(\lambda_i))$$

in (4.31) in terms of (λ_1, x_1) by considering the following three different cases for (λ_1, x_1):

1. If $\lambda_1 = 1 \in \bar{\mathcal{Q}}_x$ and $x_1 = x$, then $\lambda_2 = 0$, and $x \in [b_1, a_1]$ by (4.34). It follows from $g_2^1(\lambda_2) = g_2^1(0) = 0$ that

$$\sum_{i=1,2} \left(\lambda_i f_i(x_i) + g_i^1(\lambda_i)\right) = f_1(x) + g_1^1(1).$$

The value $f_1(x) + g_1^1(1)$ is finite because $x \in [b_1, a_1] = \operatorname{dom} f_1$.

2. If $\lambda_1 = 0 \in \bar{\mathcal{Q}}_x$ and $x_1 \in [b_1, a_1]$, then $\lambda_2 = 1$ and $x_2 = x$, where $x \in [b_2, a_2]$ by (4.35). We have $g_1^1(\lambda_1) = g_1^1(0) = 0$, and hence

$$\sum_{i=1,2} \left(\lambda_i f_i(x_i) + g_i^1(\lambda_i)\right) = f_2(x) + g_2^1(1).$$

The value $f_2(x) + g_2^1(1)$ is finite because $x \in [b_2, a_2] = \operatorname{dom} f_2$.

3. If $\lambda_1 \in \mathcal{Q}_x$ and $x_1 \in \mathcal{Z}_{\lambda_1,x}$, from the constraints, we can write λ_2 and x_2 in terms of λ_1 and x_1 respectively as

$$\begin{aligned} \lambda_2 &= 1 - \lambda_1, \\ x_2 &= \frac{x - \lambda_1 x_1}{1 - \lambda_1} = \psi_{\lambda_1,x}(x_1). \end{aligned}$$

Then

$$\sum_{i=1,2} \left(\lambda_i f_i(x_i) + g_i^1(\lambda_i)\right) = h_{\lambda_1,x}(x_1)$$

where

$$h_{\gamma,x}(z) := \gamma f_1(z) + g_1^1(\gamma) + (1-\gamma) f_2(\psi_{\gamma,x}(z)) + g_2^1(1-\gamma) \tag{4.43}$$

for all $\gamma \in (0,1)$ and $z \in \mathbb{R}$. Notice that the function $(\lambda, z) \mapsto h_{\gamma,x}(z)$ on $(0,1) \times \mathbb{R}$ is bounded from below.

Therefore, we can conclude that

$$f(x) = \min\left\{ f_1(x) + g_1^1(1), f_2(x) + g_2^1(1), \inf_{\gamma \in \mathcal{Q}_x, z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z) \right\}. \tag{4.44}$$

In (4.44), the value $f_1(x)$ is finite if and only if $1 \in \bar{\mathcal{Q}}_x$ by (4.34). Similarly, by (4.35), the value $f_2(x)$ is finite if and only if $0 \in \bar{\mathcal{Q}}_x$. Moreover, the value

$$\inf_{\gamma \in \mathcal{Q}_x, z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z) \tag{4.45}$$

is finite if and only if $\mathcal{Q}_x \neq \emptyset$. We will show that there exists a solution to the minimisation problem in (4.45).

It is straightforward to compute $f_1(x) + g_1^1(1)$ and $f_2(x) + g_2^1(1)$ in (4.44). Moreover, the problem (4.45) can be written as

$$\inf_{\gamma \in \mathcal{Q}_x} \inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z). \tag{4.46}$$

Thus, our main minimisation problem (4.31) is reduced to the two-stage minimisation problem (4.46). Observe that in each of the single-stage minimisation problems, there is only one control variable. Moreover, if $\mathcal{Q}_x = \emptyset$, then

$$\inf_{\gamma \in \mathcal{Q}_x} \inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z) = \inf \emptyset = \infty.$$

For the remainder of this section, we will always assume that

$$\mathcal{Q}_x \neq \emptyset. \tag{4.47}$$

Then $\mathcal{Q}_x$ is a subinterval of $(0,1)$.

In the remainder of this section, we will focus on finding a solution to the two-stage minimisation problem (4.46). Firstly, we will show that, for every $\gamma \in \mathcal{Q}_x$, the function $z \mapsto h_{\gamma,x}(z)$ is affine on the closed interval $\mathcal{Z}_{\gamma,x}$; see (4.52) and Proposition 4.16 below. Then we will provide a technical result in Lemma 4.17 for the convenience of later calculations. In order to find a solution to (4.46) under the assumption (4.47), we will first consider the case when $\alpha_1 = \alpha_2$. In this case, for every $\gamma \in \mathcal{Q}_x$, the values of $z \mapsto h_{\gamma,x}(z)$ will remain unchanged on $\mathcal{Z}_{\gamma,x}$; see Theorem 4.18.1 below. Thus, the control variables in (4.46) are reduced from two to one. The method for calculating the solutions to this problem is provided in Theorem 4.18.2. Secondly, we will consider the case when $\alpha_1 \neq \alpha_2$. In this case, it turns out that there exists a unique solution to (4.46). Moreover, although the problem becomes more complicated compared to the case when $\alpha_1 = \alpha_2$, we can still explicitly find the solution by considering all different cases of the given parameters. The method for deriving the unique solution to the first stage minimisation problem

$$\inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z)$$

for all $\gamma \in \mathcal{Q}_x$ is presented in Theorem 4.19. Moreover, in Theorem 4.27, we will provide a method to compute the unique solution to the second stage minimisation problem.

For convenience, for every $\gamma \in (0,1)$ and $z \in \mathbb{R}$, we define

$$\breve{f}_{\gamma,x}(z) := \gamma f_1(z) + (1-\gamma) f_2(\psi_{\gamma,x}(z)), \tag{4.48}$$

$$\breve{g}(\gamma) := g_1^1(\gamma) + g_2^1(1-\gamma). \tag{4.49}$$

From the definitions of g_1^1 and g_2^1 in (4.29), the derivatives $\breve{g}'$ and $\breve{g}''$ on $(0,1)$ are

$$\breve{g}'(\gamma) = \ln \frac{\gamma}{1-\gamma} - \ln \frac{p_1}{p_2}, \tag{4.50}$$

$$\breve{g}''(\gamma) = \frac{1}{\gamma} + \frac{1}{1-\gamma} > 0. \tag{4.51}$$

Thus $\breve{g}$ is continuous and convex on $(0,1)$. Notice that $h_{\gamma,x}(z)$ defined in (4.43) can be written as

$$h_{\gamma,x}(z) = \breve{f}_{\gamma,x}(z) + \breve{g}(\gamma). \tag{4.52}$$

For every $\gamma \in \mathcal{Q}_x$, the proposition below shows that $z \mapsto \breve{f}_{\gamma,x}(z)$ is affine on $\mathcal{Z}_{\gamma,x}$. Moreover, from (4.52), the function $z \mapsto h_{\gamma,x}(z)$ is also affine on $\mathcal{Z}_{\gamma,x}$.

Proposition 4.16. *For all $\gamma \in \mathcal{Q}_x$ and $z \in \mathcal{Z}_{\gamma,x}$, we have*

$$\breve{f}_{\gamma,x}(z) = (\alpha_1 - \alpha_2)\gamma z + \alpha_2 x + \gamma\beta_1 + (1-\gamma)\beta_2,$$

Proof. For all $\gamma \in \mathcal{Q}_x$ and $z \in \mathcal{Z}_{\gamma,x}$, it follows from Proposition 4.15 that

$$z \in [b_1, a_1] = \operatorname{dom} f_1,$$
$$\psi_{\gamma,x}(z) \in [b_2, a_2] = \operatorname{dom} f_2.$$

This means

$$f_1(z) = \alpha_1 z + \beta_1,$$
$$f_2(\psi_{\gamma,x}(z)) = \alpha_2 \psi_{\gamma,x}(z) + \beta_2 = \alpha_2 \tfrac{x-\gamma z}{1-\gamma} + \beta_2.$$

Then $\breve{f}_{\gamma,x}(z)$ defined in (4.48) can be written as

$$\begin{aligned}\breve{f}_{\gamma,x}(z) &= \gamma(\alpha_1 z + \beta_1) + (1-\gamma)\left(\alpha_2 \tfrac{x-\gamma z}{1-\gamma} + \beta_2\right)\\ &= \gamma\alpha_1 z + \gamma\beta_1 + \alpha_2(x-\gamma z) + (1-\gamma)\beta_2\\ &= (\alpha_1-\alpha_2)\gamma z + \alpha_2 x + \gamma\beta_1 + (1-\gamma)\beta_2.\end{aligned}$$

This completes the proof. □

For the convenience of later calculations, we will provide a technical result in Lemma 4.17 below. Firstly, let $I \subseteq \bar{\mathcal{Q}}_x$ be a closed interval such that

$$I \cap (0,1) \neq \emptyset. \tag{4.53}$$

Observe from $\bar{\mathcal{Q}}_x \subseteq [0,1]$ (see (4.33)) that $I \subseteq [0,1]$. Then $I = I \cap (0,1)$ if and only if I does not contain 0 and 1. We must have $\min I < 1$ and $\max I > 0$, otherwise $I \cap (0,1) = \emptyset$. For all $\gamma \in (0,1)$, we denote the point in I that is closest to γ by

$$\Gamma(\gamma; I) := \begin{cases}\gamma & \text{if } \gamma \in I,\\ \min I & \text{if } \gamma < \min I,\\ \max I & \text{if } \gamma > \max I.\end{cases} \tag{4.54}$$

Observe from the possible values of $\Gamma(\gamma;I)$ in (4.54) that $\Gamma(\gamma;I) \in I \cap (0,1)$.

Lemma 4.17. *Let $I \subseteq \bar{\mathcal{Q}}_x$ be a closed interval that satisfies* (4.53)*, and let $F : (0,1) \to \mathbb{R}$ be a continuous and differentiable function. Suppose that there exists $\gamma_0 \in (0,1)$ such that*

$$F'(\gamma) < 0 \textit{ for all } \gamma \in (0,\gamma_0), \tag{4.55}$$

$$F'(\gamma) > 0 \textit{ for all } \gamma \in (\gamma_0,1). \tag{4.56}$$

Then $\Gamma(\gamma_0;I)$ is the unique value in $I \cap (0,1)$ such that

$$F(\Gamma(\gamma_0;I)) = \inf_{\gamma \in I \cap (0,1)} F(\gamma). \tag{4.57}$$

Proof. We are going to prove this result by considering the following three cases based on the value of γ_0.

Firstly, consider the case when $\gamma_0 \in I$. From (4.55), the continuous function F is decreasing on $(0,\gamma_0]$ and hence decreasing on $I \cap (0,\gamma_0]$, where $\gamma_0 = \max(I \cap (0,\gamma_0])$. Similarly, by (4.56), the continuous function F is increasing on $[\gamma_0,1)$ and therefore increasing on $I \cap [\gamma_0,1)$, where $\gamma_0 = \min(I \cap [\gamma_0,1))$. Thus $\Gamma(\gamma_0;I) = \gamma_0$ is the unique value in $I \cap (0,1)$ such that (4.57) holds true.

Secondly, in the case when $\gamma_0 < \min I$, we have $I \cap (0,1) \subseteq (\gamma_0, 1)$. Since F is increasing on $(\gamma_0, 1)$ (by (4.56)), it is also increasing on $I \cap (0,1)$. Notice that $\min (I \cap (0,1)) = \min I$ because $0 < \gamma_0 < \min I$. Thus $\Gamma(\gamma_0; I) = \min I$ is the unique value in $I \cap (0,1)$ such that (4.57) holds true.

Thirdly, in the case when $\gamma_0 > \max I$, we have $I \cap (0,1) \subseteq (0, \gamma_0)$. From (4.55), the function F is decreasing on $(0, \gamma_0)$ and therefore decreasing on $I \cap (0,1)$. Observe from $\max I < \gamma_0 < 1$ that $\max (I \cap (0,1)) = \max I$. Thus $\Gamma(\gamma_0; I) = \max I$ is the unique value in $I \cap (0,1)$ such that (4.57) holds true. This completes the proof. $\square$

Now, we are going find a method to solve the two-stage minimisation problem (4.46). For convenience, we define $\kappa : \{b_1, b_2, a_1, a_2\} \to (0,1)$ as

$$\kappa(y) = \frac{p_1 e^{-\beta_1 - \alpha_1 y}}{p_1 e^{-\beta_1 - \alpha_1 y} + p_2 e^{-\beta_2 - \alpha_2 y}} \in (0,1), \tag{4.58}$$

where α_i and β_i are the parameters of f_i defined in (4.30) for each $i = 1, 2$. Moreover, let

$$\kappa_0 = \frac{p_1 e^{-\beta_1}}{p_1 e^{-\beta_1} + p_2 e^{-\beta_2}} \in (0,1). \tag{4.59}$$

In the case when $\alpha_1 = \alpha_2$, the slopes of the functions f_1 and f_2 on their effective domain have the same value. In such situation, we have

$$\kappa(y_1) = \kappa(y_2) = \kappa_0 \text{ for all } y_1, y_2 \in \{b_1, b_2, a_1, a_2\}.$$

In addition, the result below says that the values of $z \mapsto h_{\gamma,x}(z)$ remain unchanged on $\mathcal{Z}_{\gamma,x}$ for any $\gamma \in \mathcal{Q}_x$, and it provides a method to calculate all the solutions to the problem (4.46).

Theorem 4.18. *If $\alpha_1 = \alpha_2$, then the following two claims hold true.*

1. *For every $\gamma \in \mathcal{Q}_x$, the values of $z \mapsto h_{\gamma,x}(z)$ remain constant on $\mathcal{Z}_{\gamma,x}$.*

2. *All solutions $(\hat{\gamma}, \hat{z})$ to the two-stage minimisation problem (4.46) are of the form*
$$\hat{\gamma} = \Gamma\left(\kappa_0; \bar{\mathcal{Q}}_x\right), \qquad \hat{z} \in \mathcal{Z}_{\hat{\gamma},x},$$
where $\hat{\gamma}$ is unique but $\hat{z}$ may not.

Proof. For the convenience of the proof, we define a continuous and differentiable function $F : (0,1) \to \mathbb{R}$ as

$$F(\gamma) := \alpha_2 x + \gamma \beta_1 + (1-\gamma)\beta_2 + \breve{g}(\gamma).$$

Combining (4.52) and Proposition 4.16 together with $\alpha_1 = \alpha_2$, we have

$$h_{\gamma,x}(z) = F(\gamma) \text{ for all } \gamma \in \mathcal{Q}_x \text{ and } z \in \mathcal{Z}_{\gamma,x}. \tag{4.60}$$

This means that, for every $\gamma \in \mathcal{Q}_x$, the values of $z \mapsto h_{\gamma,x}(z)$ remain unchanged on $\mathcal{Z}_{\gamma,x}$. This completes the proof of the first claim.

For all $\gamma \in \mathcal{Q}_x$, by letting $z_{\gamma,x} \in \mathcal{Z}_{\gamma,x}$ (e.g. $z_{\gamma,x} = \min \mathcal{Z}_{\gamma,x}$), it follows that

$$h_{\gamma,x}(z_{\gamma,x}) = \inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z).$$

Then $z_{\gamma,x}$ is a solution to the first stage minimisation problem in (4.46). The second stage minimisation problem in (4.46) can be solved as follows. For every $\gamma \in (0,1)$, observe from (4.50)-(4.51) that

$$\begin{aligned} F'(\gamma) &= \beta_1 - \beta_2 + \breve{g}'(\gamma) = \beta_1 - \beta_2 + \ln \frac{\gamma}{1-\gamma} - \ln \frac{p_1}{p_2}, \\ F''(\gamma) &= \breve{g}''(\gamma) = \frac{1}{\gamma} + \frac{1}{1-\gamma} > 0. \end{aligned}$$

This means that F' is increasing on $(0,1)$. Moreover, the definition of κ_0 in (4.59) gives

$$\ln \frac{\kappa_0}{1-\kappa_0} = \ln \frac{p_1 e^{-\beta_1}}{p_2 e^{-\beta_2}} = \ln \frac{p_1}{p_2} - \beta_1 + \beta_2,$$

and hence $F'(\kappa_0) = 0$. Thus

$$\begin{aligned} F'(\gamma) < 0 \text{ for all } \gamma \in (0, \kappa_0), \\ F'(\gamma) > 0 \text{ for all } \gamma \in (\kappa_0, 1), \end{aligned}$$

By letting $\hat{\gamma} := \Gamma(\kappa_0; \bar{\mathcal{Q}}_x)$, it follows from Lemma 4.17 that $\hat{\gamma}$ is the unique value in $\bar{\mathcal{Q}}_x \cap (0,1) = \mathcal{Q}_x$ such that

$$F(\hat{\gamma}) = \inf_{\gamma \in \mathcal{Q}_x} F(\gamma). \tag{4.61}$$

Then (4.60) implies

$$h_{\hat{\gamma},x}(z_{\hat{\gamma},x}) = \inf_{\gamma \in \mathcal{Q}_x} h_{\gamma,x}(z_{\gamma,x}) = \inf_{\gamma \in \mathcal{Q}_x} \inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z).$$

For any $\hat{z} \in \mathcal{Z}_{\hat{\gamma},x}$, it follows from $h_{\hat{\gamma},x}(\hat{z}) = h_{\hat{\gamma},x}(z_{\hat{\gamma},x})$ that

$$h_{\hat{\gamma},x}(\hat{z}) = \inf_{\gamma \in \mathcal{Q}_x} \inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z).$$

Thus $(\hat{\gamma}, \hat{z})$ solves the two-stage minimisation problem (4.46). Suppose by contradiction that there exist $\hat{\gamma}' \in \mathcal{Q}_x$ and $\hat{z}' \in \mathcal{Z}_{\hat{\gamma}',x}$ such that $\hat{\gamma}' \neq \hat{\gamma}$ and

$$h_{\hat{\gamma}',x}(\hat{z}') = \inf_{\gamma \in \mathcal{Q}_x} \inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z).$$

It follows from $F(\hat{\gamma}') = h_{\hat{\gamma}',x}(\hat{z}')$ that

$$F(\hat{\gamma}') = \inf_{\gamma \in \mathcal{Q}_x} \inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z) = \inf_{\gamma \in \mathcal{Q}_x} \inf_{z \in \mathcal{Z}_{\gamma,x}} F(\gamma) = \inf_{\gamma \in \mathcal{Q}_x} F(\gamma),$$

which contradicts the fact that $\hat{\gamma}$ is the unique value in $\mathcal{Q}_x$ such that (4.61) holds true. This completes the proof of the second claim. $\square$

For the remainder of this section, we will assume that

$$\alpha_1 \neq \alpha_2.$$

This means that the slope α_1 of f_1 on $[b_1, a_1]$ and the slope α_2 of f_2 on $[b_2, a_2]$ are not the same. In this case, the minimisation problem (4.46) will be more complicated, but we can still present its solution explicitly.

For any $\gamma \in \mathcal{Q}_x$, let

$$z_{\gamma,x} := \begin{cases} \min \mathcal{Z}_{\gamma,x} & \text{if } \alpha_1 > \alpha_2, \\ \max \mathcal{Z}_{\gamma,x} & \text{if } \alpha_1 < \alpha_2. \end{cases} \tag{4.62}$$

Observe that $z_{\gamma,x}$ is the left (resp. right) endpoint of the closed interval $\mathcal{Z}_{\gamma,x}$ in the situation when $\alpha_1 > \alpha_2$ (resp. $\alpha_1 < \alpha_2$). The following result implies that, for any $\gamma \in \mathcal{Q}_x$, the quantity $z_{\gamma,x}$ defined in (4.62) is the unique solution to the first stage problem in (4.46).

Theorem 4.19. *For any $\gamma \in \mathcal{Q}_x$, the quantity $z_{\gamma,x}$ defined in* (4.62) *is the unique value in $\mathcal{Z}_{\gamma,x}$ such that*

$$h_{\gamma,x}(z_{\gamma,x}) = \inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z). \tag{4.63}$$

Proof. Fix any $\gamma \in \mathcal{Q}_x$. Combining (4.52) and Proposition 4.16, the function $z \mapsto h_{\gamma,x}(z)$ is affine on $\mathcal{Z}_{\gamma,x}$. Consider the following two cases of α_1 and α_2.

In the case when $\alpha_1 > \alpha_2$, the function $z \mapsto h_{\gamma,x}(z)$ is increasing on $\mathcal{Z}_{\gamma,x}$. Thus, the quantity $z_{\gamma,x} = \min \mathcal{Z}_{\gamma,x}$ is the unique value in $\mathcal{Z}_{\gamma,x}$ such that (4.63) holds true.

Similarly, in the case when $\alpha_1 < \alpha_2$, the function $z \mapsto h_{\gamma,x}(z)$ is decreasing on $\mathcal{Z}_{\gamma,x}$. Thus, the quantity $z_{\gamma,x} = \max \mathcal{Z}_{\gamma,x}$ is the unique value in $\mathcal{Z}_{\gamma,x}$ such that (4.63) holds true. Therefore, the result follows. $\square$

Our next objective is to find a solution to the second stage problem in (4.46). For any $\gamma \in \mathcal{Q}_x$, let

$$\widetilde{h}_x(\gamma) := h_{\gamma,x}(z_{\gamma,x}). \tag{4.64}$$

Observe from (4.52) that $h_{\gamma,x}(z_{\gamma,x}) = \breve{f}_{\gamma,x}(z_{\gamma,x}) + \breve{g}(\gamma)$ which means

$$\widetilde{h}_x(\gamma) = \breve{f}_{\gamma,x}(z_{\gamma,x}) + \breve{g}(\gamma). \tag{4.65}$$

It follows from Theorem 4.19 and (4.64) that

$$\inf_{\gamma \in \mathcal{Q}_x} \inf_{z \in \mathcal{Z}_{\gamma,x}} h_{\gamma,x}(z) = \inf_{\gamma \in \mathcal{Q}_x} h_{\gamma,x}(z_{\gamma,x}) = \inf_{\gamma \in \mathcal{Q}_x} \widetilde{h}_x(\gamma).$$

This means that the two-stage minimisation problem (4.46) is reduced to the single-stage minimisation problem:

$$\text{minimise } \widetilde{h}_x(\gamma) \text{ over } \gamma \in \mathcal{Q}_x. \tag{4.66}$$

In Theorem 4.27, we will show that there exists a unique solution to the problem (4.66), and we will provide a method to calculate this solution.

In order to concisely present the solution to (4.66) for all different cases, we shall introduce a number of shorthand notations as follows.

Firstly, under the assumption that $\alpha_1 \neq \alpha_2$, we have either $\alpha_1 < \alpha_2$ or $\alpha_1 > \alpha_2$. There are also different possibilities for the values b_1, a_1, b_2, and a_2 used for defining the effective domains of f_1 and f_2; see (4.30). We name the following six cases:

$$\begin{aligned} &\mathrm{C}^{1,1}: \alpha_1 < \alpha_2, a_1 < b_2; && \mathrm{C}^{2,1}: \alpha_1 > \alpha_2, b_1 < a_2; \\ &\mathrm{C}^{1,2}: \alpha_1 < \alpha_2, a_1 > b_2; && \mathrm{C}^{2,2}: \alpha_1 > \alpha_2, b_1 > a_2; \\ &\mathrm{C}^{1,3}: \alpha_1 < \alpha_2, a_1 = b_2; && \mathrm{C}^{2,3}: \alpha_1 > \alpha_2, b_1 = a_2. \end{aligned}$$

Notice that, in the cases $\mathrm{C}^{1,1}$, $\mathrm{C}^{1,2}$ and $\mathrm{C}^{1,3}$, the slope of f_1 on $[b_1, a_1]$ and the slope of f_2 on $[b_2, a_2]$ always satisfy $\alpha_1 < \alpha_2$. Moreover, these three cases correspond respectively to the following three different situations of a_1 and b_2: $a_1 < b_2$, $a_1 > b_2$, and $a_1 = b_2$. Similarly, in the cases $\mathrm{C}^{2,1}$, $\mathrm{C}^{2,2}$ and $\mathrm{C}^{2,3}$, we always have $\alpha_1 > \alpha_2$. In addition, these three cases correspond respectively to the following three different situations of b_1 and a_2: $b_1 < a_2$, $b_1 > a_2$, and $b_1 = a_2$.

Secondly, in the cases $\mathrm{C}^{1,1}$, $\mathrm{C}^{1,2}$, $\mathrm{C}^{2,1}$ and $\mathrm{C}^{2,2}$, we shall define two subsets of $\mathcal{Q}_x$, and these subsets will be helpful for presenting the solution to the first stage problem in (4.46). Moreover, we shall define two subsets of $\bar{\mathcal{Q}}_x$, and these subsets will be used in later calculations.

Consider the cases $\mathrm{C}^{1,1}$ and $\mathrm{C}^{1,2}$. We have $a_1 \neq b_2$. Let

$$q_x^1 := \frac{x - b_2}{a_1 - b_2}. \tag{4.67}$$

It is possible that $q_x^1 \notin \mathcal{Q}_x$. However, if $q_x^1 \in \mathcal{Q}_x$ then

$$\begin{aligned} \psi^{-1}_{q_x^1,x}(b_2) = \frac{x - (1-q_x^1)b_2}{q_x^1} &= \frac{(a_1-b_2)x - (a_1-b_2)(1-q_x^1)b_2}{(a_1-b_2)q_x^1} \\ &= \frac{(a_1-b_2)x - (a_1-x)b_2}{x-b_2} = \frac{a_1 x - a_1 b_2}{x - b_2} = a_1. \end{aligned} \tag{4.68}$$

By $\alpha_1 < \alpha_2$ and Proposition 4.15, the quantity $z_{q_x^1,x}$ defined in (4.62) can be written as

$$z_{q_x^1,x} = \max \mathcal{Z}_{q_x^1,x} = \psi^{-1}_{q_x^1,x}(b_2) = a_1.$$

The quantity q_x^1 is used to subdivide the intervals $\mathcal{Q}_x$ and $\bar{\mathcal{Q}}_x$ as follows.

- In $\mathrm{C}^{1,1}$, let

$$\begin{aligned} \mathcal{Q}^{1,1}_{1,x} &:= \Big(-\infty, q_x^1\Big] \cap \mathcal{Q}_x, & \mathcal{Q}^{1,1}_{2,x} &:= \Big[q_x^1, \infty\Big) \cap \mathcal{Q}_x, \\ \bar{\mathcal{Q}}^{1,1}_{1,x} &:= \Big(-\infty, q_x^1\Big] \cap \bar{\mathcal{Q}}_x, & \bar{\mathcal{Q}}^{1,1}_{2,x} &:= \Big[q_x^1, \infty\Big) \cap \bar{\mathcal{Q}}_x. \end{aligned}$$

- In $\mathrm{C}^{1,2}$, let

$$\begin{aligned} \mathcal{Q}^{1,2}_{1,x} &:= \Big[q_x^1, \infty\Big) \cap \mathcal{Q}_x, & \mathcal{Q}^{1,2}_{2,x} &:= \Big(-\infty, q_x^1\Big] \cap \mathcal{Q}_x, \\ \bar{\mathcal{Q}}^{1,2}_{1,x} &:= \Big[q_x^1, \infty\Big) \cap \bar{\mathcal{Q}}_x, & \bar{\mathcal{Q}}^{1,2}_{2,x} &:= \Big(-\infty, q_x^1\Big] \cap \bar{\mathcal{Q}}_x. \end{aligned}$$

The following result gives a presentation of $\gamma \mapsto z_{\gamma,x}$ on $\mathcal{Q}_x$.

Lemma 4.20. *In the case $\mathrm{C}^{1,j}$ where $j = 1, 2$, for any $\gamma \in \mathcal{Q}_x$, the value $z_{\gamma,x}$ defined in* (4.62) *can be presented as*

$$z_{\gamma,x} = \max \mathcal{Z}_{\gamma,x} = \begin{cases} \psi^{-1}_{\gamma,x}(b_2) & \text{if } \gamma \in \mathcal{Q}^{1,j}_{1,x}, \\ a_1 & \text{if } \gamma \in \mathcal{Q}^{1,j}_{2,x}. \end{cases}$$

In the situation when $\mathcal{Q}^{1,j}_{1,x} \neq \emptyset$ and $\mathcal{Q}^{1,j}_{2,x} \neq \emptyset$, we have $\mathcal{Q}^{1,j}_{1,x} \cap \mathcal{Q}^{1,j}_{2,x} = \{q_x^1\}$ and moreover $z_{\gamma,x} = \psi^{-1}_{\gamma,x}(b_2) = a_1$ for $\gamma = q_x^1$.

The proof of Lemma 4.20 above will be provided at the end of this section.

Consider the cases $\mathrm{C}^{2,1}$ and $\mathrm{C}^{2,2}$. We have $b_1 \neq a_2$. Let

$$q_x^2 := \frac{x - a_2}{b_1 - a_2}. \tag{4.69}$$

It is possible that $q_x^2 \notin \mathcal{Q}_x$. However, if $q_x^2 \in \mathcal{Q}_x$ then

$$\begin{aligned}\psi^{-1}_{q_x^2,x}(a_2) = \frac{x - (1 - q_x^2)a_2}{q_x^2} &= \frac{(b_1 - a_2)x - (b_1 - a_2)(1 - q_x^2)a_2}{(b_1 - a_2)q_x^2} \\ &= \frac{(b_1 - a_2)x - (b_1 - x)a_2}{x - a_2} = \frac{b_1 x - b_1 a_2}{x - a_2} = b_1. \end{aligned} \tag{4.70}$$

By $\alpha_1 > \alpha_2$ and Proposition 4.15, the quantity $z_{q_x^2,x}$ defined in (4.62) can be written as

$$z_{q_x^2,x} = \min \mathcal{Z}_{q_x^2,x} = \psi^{-1}_{q_x^2,x}(a_2) = b_1.$$

The quantity q_x^2 is used to subdivide the intervals $\mathcal{Q}_x$ and $\bar{\mathcal{Q}}_x$ as follows.

- In $\mathrm{C}^{2,1}$, let

$$\begin{aligned} \mathcal{Q}^{2,1}_{1,x} &= \left[q_x^2, \infty\right) \cap \mathcal{Q}_x, & \mathcal{Q}^{2,1}_{2,x} &= \left(-\infty, q_x^2\right] \cap \mathcal{Q}_x, \\ \bar{\mathcal{Q}}^{2,1}_{1,x} &= \left[q_x^2, \infty\right) \cap \bar{\mathcal{Q}}_x, & \bar{\mathcal{Q}}^{2,1}_{2,x} &= \left(-\infty, q_x^2\right] \cap \bar{\mathcal{Q}}_x. \end{aligned}$$

- In $\mathrm{C}^{2,2}$, let

$$\begin{aligned} \mathcal{Q}^{2,2}_{1,x} &= \left(-\infty, q_x^2\right] \cap \mathcal{Q}_x, & \mathcal{Q}^{2,2}_{2,x} &= \left[q_x^2, \infty\right) \cap \mathcal{Q}_x, \\ \bar{\mathcal{Q}}^{2,2}_{1,x} &= \left(-\infty, q_x^2\right] \cap \bar{\mathcal{Q}}_x, & \bar{\mathcal{Q}}^{2,2}_{2,x} &= \left[q_x^2, \infty\right) \cap \bar{\mathcal{Q}}_x. \end{aligned}$$

The lemma below gives a presentation of $\gamma \mapsto z_{\gamma,x}$ on $\mathcal{Q}_x$.

Lemma 4.21. *In the case* $\mathrm{C}^{2,j}$ *where* $j = 1, 2$*, for any* $\gamma \in \mathcal{Q}_x$*, the value* $z_{\gamma,x}$ *defined in* (4.62) *can be presented as*

$$z_{\gamma,x} = \min \mathcal{Z}_{\gamma,x} = \begin{cases} \psi^{-1}_{\gamma,x}(a_2) & \text{if } \gamma \in \mathcal{Q}^{2,j}_{1,x}, \\ b_1 & \text{if } \gamma \in \mathcal{Q}^{2,j}_{2,x}. \end{cases}$$

In the situation when $\mathcal{Q}^{2,j}_{1,x} \neq \emptyset$ *and* $\mathcal{Q}^{2,j}_{2,x} \neq \emptyset$*, we have* $\mathcal{Q}^{2,j}_{1,x} \cap \mathcal{Q}^{2,j}_{2,x} = \{q_x^2\}$ *and moreover* $z_{\gamma,x} = \psi^{-1}_{\gamma,x}(a_2) = b_1$ *for* $\gamma = q_x^2$.

The proof of Lemma 4.21 above will be provided at the end of this section. Notice that, in the case $\mathrm{C}^{i,j}$ where $i, j = 1, 2$, the intervals $\mathcal{Q}_x$ and $\bar{\mathcal{Q}}_x$ can be written as

$$\mathcal{Q}_x = \mathcal{Q}^{i,j}_{1,x} \cup \mathcal{Q}^{i,j}_{2,x},$$

$$\bar{\mathcal{Q}}_x = \bar{\mathcal{Q}}^{i,j}_{1,x} \cup \bar{\mathcal{Q}}^{i,j}_{2,x}.$$

It follows from (4.47) that $\emptyset \neq \mathcal{Q}_x \subseteq \bar{\mathcal{Q}}_x$ which implies that at least one of $\mathcal{Q}^{i,j}_{1,x}$ and $\mathcal{Q}^{i,j}_{2,x}$ is not empty, and that at least one of $\bar{\mathcal{Q}}^{i,j}_{1,x}$ and $\bar{\mathcal{Q}}^{i,j}_{2,x}$ is not empty. Moreover, we have

$$q^i_x \in \mathcal{Q}_x \Longleftrightarrow \mathcal{Q}^{i,j}_{1,x} \cap \mathcal{Q}^{i,j}_{2,x} = \left\{q^i_x\right\},$$
$$q^i_x \in \bar{\mathcal{Q}}_x \Longleftrightarrow \bar{\mathcal{Q}}^{i,j}_{1,x} \cap \bar{\mathcal{Q}}^{i,j}_{2,x} = \left\{q^i_x\right\}.$$

In addition, for each $k = 1, 2$, the definitions of $\mathcal{Q}^{i,j}_{k,x}$ and $\bar{\mathcal{Q}}^{i,j}_{k,x}$ together with $\mathcal{Q}_x = \bar{\mathcal{Q}}_x \cap (0, 1)$ (see (4.40)) imply

$$\mathcal{Q}^{i,j}_{k,x} = \bar{\mathcal{Q}}^{i,j}_{k,x} \cap (0, 1). \tag{4.71}$$

Remark 4.22. If $b_1 = a_1 > b_2 = a_2$, then we must have $b_1 < x < b_2$ to ensure $\mathcal{Q}_x \neq \emptyset$ and moreover $q^1_x = q^2_x = q^{\min}_x = q^{\max}_x$. In this special case, we can view the probability $(q^1_x, 1 - q^1_x)$ as the risk-neutral probability in a one-step friction-free model when x is the current stock price and b_1, b_2 are the discounted future stock prices.

We will provide a presentation of $\gamma \mapsto z_{\gamma,x}$ on $\mathcal{Q}_x$ for all cases in Proposition 4.23 below. For each $i, j, k = 1, 2$, let

$$u^{i,j}_k := \begin{cases} b_2 & \text{if } (i, j, k) = (1, 1, 1), (1, 2, 1), \\ a_1 & \text{if } (i, j, k) = (1, 1, 2), (1, 2, 2), \\ a_2 & \text{if } (i, j, k) = (2, 1, 1), (2, 2, 1), \\ b_1 & \text{if } (i, j, k) = (2, 1, 2), (2, 2, 2). \end{cases} \tag{4.72}$$

Observe that, for any $i, j = 1, 2$, it follows from (4.68) and (4.70) that

$$q^i_x \in \mathcal{Q}_x \Longrightarrow \psi^{-1}_{q^i_x,x}(u^{i,j}_1) = u^{i,j}_2. \tag{4.73}$$

Proposition 4.23. *For any $\gamma \in \mathcal{Q}_x$, we can present the value $z_{\gamma,x}$ defined in* (4.62) *as follows.*

1. *In the case $\mathrm{C}^{i,j}$ where $i, j = 1, 2$, the quantity $z_{\gamma,x}$ can be written as*

$$z_{\gamma,x} = \begin{cases} \psi^{-1}_{\gamma,x}\left(u^{i,j}_1\right) & \text{if } \gamma \in \mathcal{Q}^{i,j}_{1,x}, \\ u^{i,j}_2 & \text{if } \gamma \in \mathcal{Q}^{i,j}_{2,x}. \end{cases}$$

In the situation when $\mathcal{Q}^{i,j}_{1,x} \neq \emptyset$ and $\mathcal{Q}^{i,j}_{2,x} \neq \emptyset$, we have $\mathcal{Q}^{i,j}_{1,x} \cap \mathcal{Q}^{i,j}_{2,x} = \{q^i_x\}$

and moreover $z_{\gamma,x} = \psi^{-1}_{\gamma,x}(u^{i,j}_1) = u^{i,j}_2$ *for* $\gamma = q^i_x$.

2. *In the case* $\mathrm{C}^{1,3}$, *we have*

$$z_{\gamma,x} = \begin{cases} \psi^{-1}_{\gamma,x}(a_1) & \textit{if } x < a_1, \\ a_1 & \textit{if } x \geq a_1. \end{cases}$$

3. *In the case* $\mathrm{C}^{2,3}$, *we have*

$$z_{\gamma,x} = \begin{cases} a_2 & \textit{if } x \leq a_2, \\ \psi^{-1}_{\gamma,x}(a_2) & \textit{if } x > a_2. \end{cases}$$

Clearly, the function $\gamma \mapsto z_{\gamma,x}$ *is continuous on* $\mathcal{Q}_x$ *for all cases.*

Proof. Claim 1 follows directly from Lemmas 4.20-4.21 and the definition of $(u^{i,j}_k)_{i,j,k=1,2}$ in (4.72). Fix any $\gamma \in \mathcal{Q}_x$. In the case $\mathrm{C}^{1,3}$, we have $\alpha_1 < \alpha_2$ and $a_1 = b_2$. Then (4.62) and Proposition 4.15 imply

$$z_{\gamma,x} = \max \mathcal{Z}_{\gamma,x} = a_1 \wedge \psi^{-1}_{\gamma,x}(b_2) = a_1 \wedge \psi^{-1}_{\gamma,x}(a_1).$$

If $x < a_1$, then $\gamma a_1 + (1-\gamma)a_1 = a_1 > x$, in other words,

$$a_1 > \frac{x - (1-\gamma)a_1}{\gamma} = \psi^{-1}_{\gamma,x}(a_1).$$

This means $z_{\gamma,x} = \psi^{-1}_{\gamma,x}(a_1)$. Similarly, if $x \geq a_1$ then $a_1 \leq \psi^{-1}_{\gamma,x}(a_1)$ and $z_{\gamma,x} = a_1$. Thus, Claim 2 holds true. In the case $\mathrm{C}^{2,3}$, we have $\alpha_1 > \alpha_2$ and $b_1 = a_2$. It follows from (4.62) and Proposition 4.15 that

$$z_{\gamma,x} = \min \mathcal{Z}_{\gamma,x} = b_1 \vee \psi^{-1}_{\gamma,x}(a_2) = a_2 \vee \psi^{-1}_{\gamma,x}(a_2).$$

If $x \leq a_2$, then $\gamma a_2 + (1-\gamma)a_2 = a_2 \geq x$, in other words,

$$a_2 \geq \frac{x - (1-\gamma)a_2}{\gamma} = \psi^{-1}_{\gamma,x}(a_2).$$

This implies $z_{\gamma,x} = a_2$. Similarly, if $x > a_2$ then $a_2 < \psi^{-1}_{\gamma,x}(a_2)$ which means $z_{\gamma,x} = \psi^{-1}_{\gamma,x}(a_2)$. Thus, Claim 3 holds true. Notice from the presentation of $\gamma \mapsto z_{\gamma,x}$ in Claims 1-3 that $\gamma \mapsto z_{\gamma,x}$ is continuous on $\mathcal{Q}_x$ for all cases. $\square$

In Propositions 4.24 and 4.25 below, we will provide two auxiliary results for finding the value that minimises $\widetilde{h}_x(\gamma)$ over all $\gamma \in \mathcal{Q}_x$. First of all, let $c \in \{b_2, a_2\}$ and $I \subseteq \bar{\mathcal{Q}}_x$ be a closed interval such that $I \cap (0,1) \neq \emptyset$. Then we have $I \cap (0,1) \subseteq \mathcal{Q}_x$. Taking into account the presentation of $\gamma \mapsto z_{\gamma,x}$ in Proposition 4.23 and specifying c and I respectively, we have

$$z_{\gamma,x} = \psi_{\gamma,x}^{-1}(c) \text{ for all } \gamma \in I \cap (0,1) \tag{4.74}$$

in the following situations:

1. In the case $\mathrm{C}^{i,j}$ where $i,j = 1,2$, if $\mathcal{Q}_{1,x}^{i,j} \neq \emptyset$, by taking $c = u_1^{i,j} \in \{b_2, a_2\}$ and $I = \bar{\mathcal{Q}}_{1,x}^{i,j}$, we have $z_{\gamma,x} = \psi_{\gamma,x}^{-1}(u_1^{i,j}) = \psi_{\gamma,x}^{-1}(c)$ for all $\gamma \in \mathcal{Q}_{1,x}^{i,j}$, where $\mathcal{Q}_{1,x}^{i,j} = \bar{\mathcal{Q}}_{1,x}^{i,j} \cap (0,1) = I \cap (0,1)$;

2. In the case $\mathrm{C}^{1,3}$ with $x < a_1$, let $c = b_2$ and $I = \bar{\mathcal{Q}}_x$, where $b_2 = a_1$ in $\mathrm{C}^{1,3}$. It follows that $z_{\gamma,x} = \psi_{\gamma,x}^{-1}(a_1) = \psi_{\gamma,x}^{-1}(c)$ for all $\gamma \in \mathcal{Q}_x = I \cap (0,1)$;

3. In the case $\mathrm{C}^{2,3}$ with $x > a_2$, we take $c = a_2$ and $I = \bar{\mathcal{Q}}_x$, which gives $z_{\gamma,x} = \psi_{\gamma,x}^{-1}(a_2) = \psi_{\gamma,x}^{-1}(c)$ for all $\gamma \in \mathcal{Q}_x = I \cap (0,1)$.

As long as (4.74) holds true, for any $\gamma \in I \cap (0,1)$, the value $\gamma z_{\gamma,x}$ is

$$\gamma z_{\gamma,x} = \gamma \psi_{\gamma,x}^{-1}(c) = \gamma \frac{x - (1-\gamma)c}{\gamma} = x - (1-\gamma)c = \gamma c + x - c.$$

Then it follows from Proposition 4.16 that

$$\begin{aligned} \breve{f}_{\gamma,x}(z_{\gamma,x}) &= (\alpha_1 - \alpha_2)\gamma z_{\gamma,x} + \alpha_2 x + \gamma\beta_1 + (1-\gamma)\,\beta_2 \\ &= (\alpha_1 - \alpha_2)(\gamma c + x - c) + (\beta_1 - \beta_2)\gamma + \alpha_2 x + \beta_2 \\ &= \gamma[(\alpha_1 - \alpha_2)c + \beta_1 - \beta_2] + \alpha_1(x - c) + \alpha_2 c + \beta_2; \end{aligned} \tag{4.75}$$

see (4.48) for the definition of $z \mapsto \breve{f}_{\gamma,x}(z)$. Combining this with (4.65), the function $\widetilde{h}_x$ on $I \cap (0,1)$ can be written as

$$\widetilde{h}_x\,(\gamma) = \gamma[(\alpha_1 - \alpha_2)c + \beta_1 - \beta_2] + \alpha_1\,(x - c) + \alpha_2 c + \beta_2 + \breve{g}(\gamma).$$

Moreover, the following result shows that $\Gamma(\kappa(c); I)$ is the unique quantity that minimises $\widetilde{h}_x(\gamma)$ over all $\gamma \in I \cap (0,1)$; see (4.58) for the definition of $\kappa(c)$ and (4.54) for the definition of $\Gamma(\kappa(c); I)$.

Proposition 4.24. *Let $c \in \{b_2, a_2\}$ and $I \subseteq \bar{\mathcal{Q}}_x$ be a closed interval such that $I \cap (0,1) \neq \emptyset$. If (4.74) holds true, then $\Gamma(\kappa(c); I)$ is the unique value in $I \cap (0,1)$ such that*

$$\widetilde{h}_x\left(\Gamma(\kappa(c); I)\right) = \inf_{\gamma \in I \cap (0,1)} \widetilde{h}_x\left(\gamma\right). \tag{4.76}$$

Proof. For convenience, we define $F : (0,1) \to \mathbb{R}$ as

$$F(\gamma) = \gamma[(\alpha_1 - \alpha_2)c + \beta_1 - \beta_2] + \alpha_1 (x - c) + \alpha_2 c + \beta_2 + \breve{g}(\gamma).$$

Notice that F is continuous and differentiable on $(0,1)$. Moreover, we have $F = \widetilde{h}_x$ on $I \cap (0,1)$. For every $\gamma \in (0,1)$, it follows from (4.50)-(4.51) that

$$\begin{aligned} F'(\gamma) &= (\alpha_1 - \alpha_2)c + \beta_1 - \beta_2 + \breve{g}'(\gamma) \\ &= (\alpha_1 - \alpha_2)c + \beta_1 - \beta_2 + \ln \frac{\gamma}{1-\gamma} - \ln \frac{p_1}{p_2}, \\ F''(\gamma) &= \breve{g}''(\gamma) = \frac{1}{\gamma} + \frac{1}{1-\gamma} > 0. \end{aligned} \tag{4.77}$$

The definition of $\kappa(c)$ in (4.58) gives

$$\ln \frac{\kappa(c)}{1-\kappa(c)} = \ln \frac{p_1 e^{-\beta_1 - \alpha_1 c}}{p_2 e^{-\beta_2 - \alpha_2 c}} = \ln \frac{p_1}{p_2} - \beta_1 + \beta_2 - (\alpha_1 - \alpha_2)c.$$

Then (4.77) implies that $F'(\kappa(c)) = 0$. Combining this with the fact that F' is increasing on $(0,1)$, it follows that

$$\begin{aligned} F'(\gamma) &< 0 \text{ for all } \gamma \in (0, \kappa(c)), \\ F'(\gamma) &> 0 \text{ for all } \gamma \in (\kappa(c), 1). \end{aligned}$$

Then Lemma 4.17 implies that $\Gamma(\kappa(c); I)$ is the unique value in $I \cap (0,1)$ such that

$$F\left(\Gamma(\kappa(c); I)\right) = \inf_{\gamma \in I \cap (0,1)} F(\gamma).$$

Since $F = \widetilde{h}_x$ on $I \cap (0,1)$, the quantity $\Gamma(\kappa(c); I)$ is also the unique value in $I \cap (0,1)$ such that (4.76) holds true. This completes the proof. □

Proposition 4.25 below is similar to Proposition 4.24 but with a different assumption. These two propositions will be used to prove Theorem 4.27 which will provide a method to compute the unique solution of the minimisation problem (4.66). Let $c \in \{b_1, a_1\}$ and $I \subseteq \bar{\mathcal{Q}}_x$ be a closed interval such that $I \cap (0,1) \neq \emptyset$, where $I \cap (0,1) \subseteq \mathcal{Q}_x$. Taking into account the presentation of $\gamma \mapsto z_{\gamma,x}$ in Proposition 4.23 and specifying c and I respectively, we have

$$z_{\gamma,x} = c \text{ for all } \gamma \in I \cap (0,1). \tag{4.78}$$

in the following situations:

1. In the case $\mathrm{C}^{i,j}$ where $i,j = 1,2$, if $\mathcal{Q}^{i,j}_{2,x} \neq \emptyset$, by taking $c = u^{i,j}_2 \in \{b_1, a_1\}$ and $I = \bar{\mathcal{Q}}^{i,j}_{2,x}$, it follows that $z_{\gamma,x} = u^{i,j}_2 = c$ for all $\gamma \in \mathcal{Q}^{i,j}_{2,x}$, where

$\mathcal{Q}^{i,j}_{2,x} = \bar{\mathcal{Q}}^{i,j}_{2,x} \cap (0,1) = I \cap (0,1)$;

2. In the case $\mathrm{C}^{1,3}$ with $x \geq a_1$, let $c = a_1$ and $I = \bar{\mathcal{Q}}_x$, which gives $z_{\gamma,x} = a_1 = c$ for all $\gamma \in \mathcal{Q}_x = I \cap (0,1)$;

3. In the case $\mathrm{C}^{2,3}$ with $x \leq a_2$, we take $c = b_1$ and $I = \bar{\mathcal{Q}}_x$, where $b_1 = a_2$ in $\mathrm{C}^{2,3}$. It follows that $z_{\gamma,x} = a_2 = c$ for all $\gamma \in \mathcal{Q}_x = I \cap (0,1)$.

As long as (4.78) holds true, for any $\gamma \in I \cap (0,1)$, it follows from Proposition 4.16 that

$$\begin{aligned}
\breve{f}_{\gamma,x}(z_{\gamma,x}) &= (\alpha_1 - \alpha_2)\gamma z_{\gamma,x} + \alpha_2 x + \gamma\beta_1 + (1-\gamma)\,\beta_2 \\
&= (\alpha_1 - \alpha_2)\gamma c + \alpha_2 x + \gamma\beta_1 + (1-\gamma)\,\beta_2 \\
&= \gamma[(\alpha_1 - \alpha_2)c + \beta_1 - \beta_2] + \alpha_2 x + \beta_2. \qquad (4.79)
\end{aligned}$$

Combining this with (4.65), it follows that

$$\widetilde{h}_x(\gamma) = \gamma[(\alpha_1 - \alpha_2)c + \beta_1 - \beta_2] + \alpha_2 x + \beta_2 + \breve{g}(\gamma).$$

Moreover, the following result shows that $\Gamma(\kappa(c); I)$ is the unique quantity that minimises $\widetilde{h}_x(\gamma)$ over all $\gamma \in I \cap (0,1)$.

Proposition 4.25. *Let $c \in \{b_1, a_1\}$ and $I \subseteq \bar{\mathcal{Q}}_x$ be a closed interval such that $I \cap (0,1) \neq \emptyset$. If (4.78) holds true, then $\Gamma(\kappa(c); I)$ is the unique value in $I \cap (0,1)$ such that*

$$\widetilde{h}_x\left(\Gamma(\kappa(c); I)\right) = \inf_{\gamma \in I \cap (0,1)} \widetilde{h}_x\left(\gamma\right). \qquad (4.80)$$

Proof. For convenience, we define $F : (0,1) \to \mathbb{R}$ as

$$F(\gamma) = \gamma[(\alpha_1 - \alpha_2)c + \beta_1 - \beta_2] + \alpha_2 x + \beta_2 + \breve{g}(\gamma).$$

Clearly, the function F is continuous and differentiable on $(0,1)$, and moreover $F = \widetilde{h}_x$ on $I \cap (0,1)$. For any $\gamma \in (0,1)$, it follows from $\breve{g}'(\gamma)$ and $\breve{g}''(\gamma)$ calculated in (4.50)-(4.51) that

$$\begin{aligned}
F'(\gamma) &= (\alpha_1 - \alpha_2)c + \beta_1 - \beta_2 + \breve{g}'(\gamma) \\
&= (\alpha_1 - \alpha_2)c + \beta_1 - \beta_2 + \ln\frac{\gamma}{1-\gamma} - \ln\frac{p_1}{p_2}, \qquad (4.81) \\
F''(\gamma) &= \breve{g}''(\gamma) = \frac{1}{\gamma} + \frac{1}{1-\gamma} > 0.
\end{aligned}$$

This means that F' is increasing on $I \cap (0,1)$. The definition of $\kappa(c)$ in (4.58) gives

$$\ln \frac{\kappa(c)}{1-\kappa(c)} = \ln \frac{p_1 e^{-\beta_1 - \alpha_1 c}}{p_2 e^{-\beta_2 - \alpha_2 c}} = \ln \frac{p_1}{p_2} - \beta_1 + \beta_2 - (\alpha_1 - \alpha_2)c.$$

Then (4.81) implies that $F'(\kappa(c)) = 0$. Combining this with the fact that F' is increasing on $(0,1)$, it follows that

$$F'(\gamma) < 0 \text{ for all } \gamma \in (0, \kappa(c)),$$
$$F'(\gamma) > 0 \text{ for all } \gamma \in (\kappa(c), 1).$$

Then Lemma 4.17 implies that $\Gamma(\kappa(c); I)$ is the unique value in $I \cap (0,1)$ such that

$$F(\Gamma(\kappa(c); I)) = \inf_{\gamma \in I \cap (0,1)} F(\gamma).$$

Combining this with $F = \widetilde{h}_x$ on $I \cap (0,1)$, the quantity $\Gamma(\kappa(c); I)$ is also the unique value in $I \cap (0,1)$ such that (4.80) holds true. This completes the proof. □

The proposition below gives the continuity and convexity of $\widetilde{h}_x$. This result will be used in Theorem 4.27 to prove the uniqueness of solution to the minimisation problem (4.66).

Proposition 4.26. *The function $\widetilde{h}_x$ is continuous and convex on $\mathcal{Q}_x$.*

Proof. From (4.65) and the continuity and the convexity of $\breve{g}$, the function $\widetilde{h}_x$ is continuous and convex on $\mathcal{Q}_x$ as long as $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is continuous and convex on $\mathcal{Q}_x$. From Proposition 4.16, the function $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ on $\mathcal{Q}_x$ can be presented as

$$\breve{f}_{\gamma,x}(z_{\gamma,x}) = (\alpha_1 - \alpha_2)\gamma z_{\gamma,x} + \alpha_2 x + \gamma\beta_1 + (1-\gamma)\beta_2.$$

By Proposition 4.23, the function $\gamma \mapsto z_{\gamma,x}$ is continuous on $\mathcal{Q}_x$. Thus, the function $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is continuous on $\mathcal{Q}_x$. We are going to show that $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is always convex on $\mathcal{Q}_x$ by considering the case $\mathrm{C}^{i,j}$ for each $i = 1,2$ and $j = 1,2,3$.

Firstly, we consider the case $\mathrm{C}^{i,j}$ where $i,j = 1,2$. Notice that at least one of $\mathcal{Q}^{i,j}_{1,x}$ and $\mathcal{Q}^{i,j}_{2,x}$ is not empty because $\mathcal{Q}^{i,j}_{1,x} \cup \mathcal{Q}^{i,j}_{2,x} = \mathcal{Q}_x \neq \emptyset$. In the situation when $\mathcal{Q}^{i,j}_{1,x} \neq \emptyset$, the condition (4.74) holds true for $c = u^{i,j}_1$ and $I = \bar{\mathcal{Q}}^{i,j}_{1,x}$; see Item 1 in the list following (4.74). For any $\gamma \in \mathcal{Q}^{i,j}_{1,x} = I \cap (0,1)$, we have from (4.75) that

$$\breve{f}_{\gamma,x}(z_{\gamma,x}) = \rho_1\gamma + \alpha_1(x - u^{i,j}_1) + \alpha_2 u^{i,j}_1 + \beta_2$$

where

$$\rho_1 := (\alpha_1 - \alpha_2)u_1^{i,j} + \beta_1 - \beta_2.$$

This means that $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is affine on $\mathcal{Q}_{1,x}^{i,j}$ with slope ρ_1. Similarly, in the situation when $\mathcal{Q}_{2,x}^{i,j} \neq \emptyset$, the condition (4.78) holds true for $c = u_2^{i,j}$ and $I = \bar{\mathcal{Q}}_{2,x}^{i,j}$; see Item 1 in the list following (4.78). For any $\gamma \in \mathcal{Q}_{2,x}^{i,j} = I \cap (0,1)$, it follows from (4.79) that

$$\breve{f}_{\gamma,x}(z_{\gamma,x}) = \rho_2\gamma + \alpha_2 x + \beta_2$$

where

$$\rho_2 := (\alpha_1 - \alpha_2)u_2^{i,j} + \beta_1 - \beta_2.$$

This means that $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is affine on $\mathcal{Q}_{2,x}^{i,j}$ with slope ρ_2. Notice that, if either $\mathcal{Q}_{1,x}^{i,j} = \emptyset$ or $\mathcal{Q}_{2,x}^{i,j} = \emptyset$, then $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is affine and hence convex on $\mathcal{Q}_x$. Suppose now that $\mathcal{Q}_{1,x}^{i,j} \neq \emptyset$ and $\mathcal{Q}_{2,x}^{i,j} \neq \emptyset$. Then $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is continuous and piecewise linear on $\mathcal{Q}_x$ with two segments. The difference between ρ_1 and ρ_2 is

$$\rho_1 - \rho_2 = (\alpha_1 - \alpha_2)\left(u_1^{i,j} - u_2^{i,j}\right),$$

where the values of the given parameters $\alpha_1, \alpha_2, u_1^{i,j}, u_2^{i,j}$ satisfy:

$$\begin{aligned}
(i,j) = (1,1): &\quad \alpha_1 - \alpha_2 < 0,\ \ u_1^{1,1} - u_2^{1,1} = b_2 - a_1 > 0;\\
(i,j) = (1,2): &\quad \alpha_1 - \alpha_2 < 0,\ \ u_1^{1,2} - u_2^{1,2} = b_2 - a_1 < 0;\\
(i,j) = (2,1): &\quad \alpha_1 - \alpha_2 > 0,\ \ u_1^{2,1} - u_2^{2,1} = a_2 - b_1 > 0;\\
(i,j) = (2,2): &\quad \alpha_1 - \alpha_2 > 0,\ \ u_1^{2,2} - u_2^{2,2} = a_2 - b_1 < 0.
\end{aligned}$$

If $(i,j) = (1,1), (2,2)$, then $\rho_1 - \rho_2 < 0$. Combining this with

$$\mathcal{Q}_{1,x}^{i,j} = \left(-\infty, q_x^i\right] \cap \mathcal{Q}_x, \qquad\qquad \mathcal{Q}_{2,x}^{i,j} = \left[q_x^i, \infty\right) \cap \mathcal{Q}_x,$$

the function $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ convex on $\mathcal{Q}_x$. Similarly, if $(i,j) = (1,2), (2,1)$, then $\rho_1 - \rho_2 > 0$. It follows from

$$\mathcal{Q}_{1,x}^{i,j} = \left[q_x^i, \infty\right) \cap \mathcal{Q}_x, \qquad\qquad \mathcal{Q}_{2,x}^{i,j} = \left(-\infty, q_x^i\right] \cap \mathcal{Q}_x$$

that $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is convex on $\mathcal{Q}_x$.

Secondly, we consider the cases $\mathrm{C}^{1,3}$ and $\mathrm{C}^{2,3}$. In both cases, we are going to show that $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is an affine function on $\mathcal{Q}_x$. Fix any $\gamma \in \mathcal{Q}_x$. In the case $\mathrm{C}^{1,3}$ with $x < a_1$, the condition (4.74) holds true for $c = b_2$ and

$I = \bar{\mathcal{Q}}_x$; see Item 2 in the list following (4.74). Then $\gamma \in \mathcal{Q}_x = I \cap (0,1)$, and (4.75) implies that

$$\breve{f}_{\gamma,x}(z_{\gamma,x}) = \gamma[(\alpha_1 - \alpha_2)b_2 + \beta_1 - \beta_2] + \alpha_1(x - b_2) + \alpha_2 b_2 + \beta_2.$$

Similarly, when $x \geq a_1$, the condition (4.78) holds true for $c = a_1$ and $I = \bar{\mathcal{Q}}_x$; see Item 2 in the list following (4.78). Then $\gamma \in \mathcal{Q}_x = I \cap (0,1)$, and (4.79) implies that

$$\breve{f}_{\gamma,x}(z_{\gamma,x}) = \gamma[(\alpha_1 - \alpha_2)a_1 + \beta_1 - \beta_2] + \alpha_2 x + \beta_2.$$

In the case $\mathrm{C}^{2,3}$ with $x \leq a_2$, the condition (4.78) holds true for $c = b_1$ and $I = \bar{\mathcal{Q}}_x$; see Item 3 in the list following (4.78). Then $\gamma \in \mathcal{Q}_x = I \cap (0,1)$, and (4.79) implies that

$$\breve{f}_{\gamma,x}(z_{\gamma,x}) = \gamma[(\alpha_1 - \alpha_2)b_1 + \beta_1 - \beta_2] + \alpha_2 x + \beta_2.$$

Similarly, when $x > a_2$, the condition (4.74) holds true for $c = a_2$ and $I = \bar{\mathcal{Q}}_x$; see Item 3 in the list following (4.74). Then $\gamma \in \mathcal{Q}_x = I \cap (0,1)$, and (4.75) implies that

$$\breve{f}_{\gamma,x}(z_{\gamma,x}) = \gamma[(\alpha_1 - \alpha_2)a_2 + \beta_1 - \beta_2] + \alpha_1(x - a_2) + \alpha_2 a_2 + \beta_2.$$

Thus, the function $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is affine and hence convex on $\mathcal{Q}_x$ for the cases $\mathrm{C}^{1,3}$ and $\mathrm{C}^{2,3}$.

The conclusion is that $\gamma \mapsto \breve{f}_{\gamma,x}(z_{\gamma,x})$ is always continuous and convex on $\mathcal{Q}_x$. This completes the proof. □

The following theorem shows that there exists a unique solution to the minimisation problem (4.66), and it provides a method to calculate this solution.

Theorem 4.27. *There exists a unique value $\hat{\gamma}_x \in \mathcal{Q}_x$ such that*

$$\widetilde{h}_x(\hat{\gamma}_x) = \inf_{\gamma \in \mathcal{Q}_x} \widetilde{h}_x(\gamma), \tag{4.82}$$

and $\hat{\gamma}_x$ is given as follows.

1. *In the case $\mathrm{C}^{i,j}$ where $i, j = 1, 2$, the value $\hat{\gamma}_x$ can be presented as*

$$\hat{\gamma}_x = \arg\min \left\{ \widetilde{h}_x(\gamma) \,\middle|\, \gamma = \Gamma\left(\kappa\left(u_k^{i,j}\right); \bar{\mathcal{Q}}_{k,x}^{i,j}\right), \mathcal{Q}_{k,x}^{i,j} \neq \emptyset, k = 1, 2 \right\}.$$

2. *In the case $\mathrm{C}^{i,3}$ where $i = 1, 2$, we have $\hat{\gamma}_x = \Gamma\left(\kappa\left(a_i\right); \bar{\mathcal{Q}}_x\right)$.*

Proof. In the case $\mathrm{C}^{i,j}$ where $i,j=1,2$, from Proposition 4.23.1 and (4.71), for any $\gamma\in\mathcal{Q}_x$, the value $z_{\gamma,x}$ can be presented as

$$z_{\gamma,x}=\begin{cases}\psi_{\gamma,x}^{-1}\left(u_1^{i,j}\right) & \text{if } \gamma\in\mathcal{Q}_{1,x}^{i,j}=\bar{\mathcal{Q}}_{1,x}^{i,j}\cap(0,1),\\ u_2^{i,j} & \text{if } \gamma\in\mathcal{Q}_{2,x}^{i,j}=\bar{\mathcal{Q}}_{2,x}^{i,j}\cap(0,1).\end{cases}$$

If $\bar{\mathcal{Q}}_{1,x}^{i,j}\cap(0,1)\neq\emptyset$, then (4.74) holds true for $c=u_1^{i,j}$ and $I=\bar{\mathcal{Q}}_{1,x}^{i,j}$; see Item 1 in the list following (4.74). Then Proposition 4.24 implies that $\Gamma(\kappa(u_1^{i,j});\bar{\mathcal{Q}}_{1,x}^{i,j})$ is the unique value in $\mathcal{Q}_{1,x}^{i,j}$ such that

$$\widetilde{h}_x\left(\Gamma\left(\kappa\left(u_1^{i,j}\right);\bar{\mathcal{Q}}_{1,x}^{i,j}\right)\right)=\inf\left\{\widetilde{h}_x(\gamma)\,\middle|\,\gamma\in\mathcal{Q}_{1,x}^{i,j}\right\}.$$

Similarly, if $\bar{\mathcal{Q}}_{2,x}^{i,j}\cap(0,1)\neq\emptyset$, then (4.78) holds true for $c=u_2^{i,j}$ and $I=\bar{\mathcal{Q}}_{2,x}^{i,j}$; see Item 1 in the list following (4.78). Then Proposition 4.25 implies that $\Gamma(\kappa(u_2^{i,j});\bar{\mathcal{Q}}_{2,x}^{i,j})$ is the unique value in $\mathcal{Q}_{2,x}^{i,j}$ such that

$$\widetilde{h}_x\left(\Gamma\left(\kappa\left(u_2^{i,j}\right);\bar{\mathcal{Q}}_{2,x}^{i,j}\right)\right)=\inf\left\{\widetilde{h}_x(\gamma)\,\middle|\,\gamma\in\mathcal{Q}_{2,x}^{i,j}\right\}.$$

Since $\mathcal{Q}_{1,x}^{i,j}\cup\mathcal{Q}_{2,x}^{i,j}=\mathcal{Q}_x$, the quantity $\hat{\gamma}_x$ presented in Theorem 4.27.1 satisfies (4.82). We are going to prove the uniqueness of $\hat{\gamma}_x$. Suppose by contradiction that $\hat{\gamma}_x$ is not unique. Then $\mathcal{Q}_{1,x}^{i,j}\neq\emptyset$ and $\mathcal{Q}_{2,x}^{i,j}\neq\emptyset$, and moreover

$$\Gamma\left(\kappa\left(u_1^{i,j}\right);\bar{\mathcal{Q}}_{1,x}^{i,j}\right)\neq\Gamma\left(\kappa\left(u_2^{i,j}\right);\bar{\mathcal{Q}}_{2,x}^{i,j}\right)$$

and

$$\widetilde{h}_x\left(\Gamma\left(\kappa\left(u_1^{i,j}\right);\bar{\mathcal{Q}}_{1,x}^{i,j}\right)\right)=\widetilde{h}_x\left(\Gamma\left(\kappa\left(u_2^{i,j}\right);\bar{\mathcal{Q}}_{2,x}^{i,j}\right)\right).$$

Combining this with the fact that $\Gamma(\kappa(u_k^{i,j});\bar{\mathcal{Q}}_{k,x}^{i,j})$, where $k=1,2$, is the unique value in $\mathcal{Q}_{k,x}^{i,j}$ that minimises $\widetilde{h}_x(\gamma)$ over all $\gamma\in\mathcal{Q}_{k,x}^{i,j}$, there exists γ' between $\Gamma(\kappa(u_1^{i,j});\bar{\mathcal{Q}}_{1,x}^{i,j})$ and $\Gamma(\kappa(u_2^{i,j});\bar{\mathcal{Q}}_{2,x}^{i,j})$ such that

$$\widetilde{h}_x(\gamma')>\widetilde{h}_x\left(\Gamma\left(\kappa\left(u_1^{i,j}\right);\bar{\mathcal{Q}}_{1,x}^{i,j}\right)\right)=\widetilde{h}_x\left(\Gamma\left(\kappa\left(u_2^{i,j}\right);\bar{\mathcal{Q}}_{2,x}^{i,j}\right)\right).$$

However, this contradicts the convexity of $\widetilde{h}_x$ (established in Proposition 4.26). This completes the proof of the uniqueness of $\hat{\gamma}_x$.

Consider the case $\mathrm{C}^{1,3}$. If $x<a_1$, then (4.74) holds true for $c=b_2=a_1$ and $I=\bar{\mathcal{Q}}_x$; see Item 2 in the list following (4.74). Then Proposition 4.24 implies that $\hat{\gamma}_x=\Gamma(\kappa(a_1);\bar{\mathcal{Q}}_x)$ is the unique value in $\mathcal{Q}_x$ such that (4.82) holds true. If $x\geq a_1$, then (4.78) holds true for $c=a_1$ and $I=\bar{\mathcal{Q}}_x$; see Item 2 in the list following (4.78). Then Proposition 4.25 implies that $\hat{\gamma}_x=\Gamma(\kappa(a_1);\bar{\mathcal{Q}}_x)$ is the unique value in $\mathcal{Q}_x$ such that (4.82) holds true.

Consider the case $\mathrm{C}^{2,3}$. If $x > a_2$, then (4.74) holds true for $c = a_2$ and $I = \bar{\mathcal{Q}}_x$; see Item 3 in the list following (4.74). Then Proposition 4.24 implies that $\hat{\gamma}_x = \Gamma(\kappa(a_2); \bar{\mathcal{Q}}_x)$ is the unique value in $\mathcal{Q}_x$ such that (4.82) holds true. If $x \leq a_2$, then (4.78) holds true for $c = b_1 = a_2$ and $I = \bar{\mathcal{Q}}_x$; see Item 3 in the list following (4.78). Then $\hat{\gamma}_x = \Gamma(\kappa(a_2); \bar{\mathcal{Q}}_x)$ is the unique value in $\mathcal{Q}_x$ such that (4.82) holds true (Proposition 4.25). This completes the proof. □

This section ends with the proofs of Lemmas 4.14, 4.20, and 4.21.

Proof of Lemma 4.14. From (4.33) and the comments following it, we have

$$\begin{aligned}\bar{\mathcal{Q}}_x = \{\gamma \in [0,1] \,|\, \exists\, x_1 \in [b_1, a_1], x_2 \in [b_2, a_2] : \gamma x_1 + (1-\gamma)\, x_2 = x\} \\ \neq \emptyset.\end{aligned} \tag{4.83}$$

Observe from Lemma A.7 that $\bar{\mathcal{Q}}_x$ is convex. Thus $\bar{\mathcal{Q}}_x$ is a subinterval of $[0,1]$.

For the convenience of later calculations, we define

$$\mu_x(x_1, x_2) := \frac{x - x_2}{x_1 - x_2} \text{ for all } x_1, x_2 \in \mathbb{R} \text{ such that } x_1 \neq x_2,$$

For any $x_2 \in \mathbb{R}$, the derivative $\frac{\partial}{\partial x_1}\mu_x(x_1, x_2)$ is

$$\frac{\partial}{\partial x_1}\mu_x(x_1, x_2) = \frac{x_2 - x}{(x_1 - x_2)^2} \text{ for all } x_1 \in \mathbb{R}\backslash\{x_2\}.$$

Similarly, for any $x_1 \in \mathbb{R}$, the derivative $\frac{\partial}{\partial x_2}\mu_x(x_1, x_2)$ is

$$\frac{\partial}{\partial x_2}\mu_x(x_1, x_2) = \frac{x - x_1}{(x_1 - x_2)^2} \text{ for all } x_2 \in \mathbb{R}\backslash\{x_1\}.$$

We are going to show that

$$\min \bar{\mathcal{Q}}_x = q_x^{\min} \leq q_x^{\max} = \max \bar{\mathcal{Q}}_x \tag{4.84}$$

always holds true by considering the following three cases of the relationship between x and $[b_2, a_2]$: $x < b_2$, $a_2 < x$, and $b_2 \leq x \leq a_2$.

Firstly, we consider the case when $x < b_2$. To ensure (4.32), we must have $b_1 \leq x$, and hence $b_1 \leq x < b_2$. Then, for any $x_1 > x$, it follows that

$$\gamma x_1 + (1-\gamma)\, x_2 > x \text{ for all } \gamma \in [0,1], x_2 \in [b_2, a_2].$$

Combining this with (4.83), the family $\bar{\mathcal{Q}}_x$ can be written as

$$\bar{\mathcal{Q}}_x = \{\gamma \in [0,1] \,|\, \exists\, x_1 \in [b_1, a_1 \wedge x], x_2 \in [b_2, a_2] : \gamma x_1 + (1-\gamma)\, x_2 = x\}$$

$$= \left\{ \gamma \in [0,1] \middle| \exists\, x_1 \in [b_1, a_1 \wedge x], x_2 \in [b_2, a_2] : \gamma = \frac{x - x_2}{x_1 - x_2} \right\},$$

where $(a_1 \wedge x) < b_2$ which means that there is no overlap between the intervals $[b_1, a_1 \wedge x]$ and $[b_2, a_2]$. Observe that

$$\mu_x(x_1, x_2) = \frac{x - x_2}{x_1 - x_2} \in (0,1] \text{ for all } x_1 \in [b_1, a_1 \wedge x], x_2 \in [b_2, a_2].$$

Then $\bar{\mathcal{Q}}_x$ can be presented as

$$\bar{\mathcal{Q}}_x = \{\mu_x(x_1, x_2) \,|x_1 \in [b_1, a_1 \wedge x], x_2 \in [b_2, a_2]\}.$$

For any $x_2 \in [b_2, a_2]$, it follows from $x_2 - x > x_2 - b_2 \geq 0$ that

$$\frac{\partial}{\partial x_1} \mu_x(x_1, x_2) = \frac{x_2 - x}{(x_1 - x_2)^2} > 0 \text{ for all } x_1 \in [b_1, a_1 \wedge x].$$

This implies that the function $x_1 \mapsto \mu_x(x_1, x_2)$ is increasing on $[b_1, a_1 \wedge x]$. For any $x_1 \in [b_1, a_1 \wedge x]$, we have from $x - x_1 \geq x - (a_1 \wedge x) \geq 0$ that

$$\frac{\partial}{\partial x_2} \mu_x(x_1, x_2) = \frac{x - x_1}{(x_1 - x_2)^2} \geq 0 \text{ for all } x_2 \in [b_2, a_2].$$

This means that $x_2 \mapsto \mu_x(x_1, x_2)$ is nondecreasing on $[b_2, a_2]$. The conclusion is that

$$\begin{aligned} q_x^{\min} = \mu_x(b_1, b_2) &= \min_{x_1 \in [b_1, a_1 \wedge x], x_2 \in [b_2, a_2]} \mu_x(x_1, x_2) = \min \bar{\mathcal{Q}}_x, \\ q_x^{\max} = \mu_x(a_1 \wedge x, a_2) &= \max_{x_1 \in [b_1, a_1 \wedge x], x_2 \in [b_2, a_2]} \mu_x(x_1, x_2) = \max \bar{\mathcal{Q}}_x. \end{aligned}$$

Clearly, we have $q_x^{\min} \leq q_x^{\max}$. Therefore (4.84) holds true.

Secondly, we consider the case when $a_2 < x$. To ensure (4.32), we must have $x \leq a_1$, which means $a_2 < x \leq a_1$. Then we have for any $x_1 < x$ that

$$\gamma x_1 + (1 - \gamma)\, x_2 < x \text{ for all } \gamma \in [0,1], x_2 \in [b_2, a_2].$$

Combining this with the formulation of $\bar{\mathcal{Q}}_x$ in (4.83), it follows that

$$\begin{aligned} \bar{\mathcal{Q}}_x &= \{\gamma \in [0,1] \,|\exists\, x_1 \in [b_1 \vee x, a_1], x_2 \in [b_2, a_2] : \gamma x_1 + (1 - \gamma)\, x_2 = x\} \\ &= \left\{ \gamma \in [0,1] \middle| \exists\, x_1 \in [b_1 \vee x, a_1], x_2 \in [b_2, a_2] : \gamma = \frac{x - x_2}{x_1 - x_2} \right\}, \end{aligned}$$

where $a_2 < (b_1 \vee x)$ which means that there is no overlap between the intervals

$[b_1 \vee x, a_1]$ and $[b_2, a_2]$. Notice that

$$\mu_x(x_1, x_2) = \frac{x - x_2}{x_1 - x_2} \in (0, 1] \text{ for all } x_1 \in [b_1 \vee x, a_1], x_2 \in [b_2, a_2]$$

Then $\bar{\mathcal{Q}}_x$ can be written as

$$\bar{\mathcal{Q}}_x = \{\mu_x(x_1, x_2) \,|x_1 \in [b_1 \vee x, a_1], x_2 \in [b_2, a_2]\} .$$

For any $x_2 \in [b_2, a_2]$, we have from $x_2 - x < x_2 - a_2 \leq 0$ that

$$\frac{\partial}{\partial x_1}\mu_x(x_1, x_2) = \frac{x_2 - x}{(x_1 - x_2)^2} < 0 \text{ for all } x_1 \in [b_1 \vee x, a_1].$$

This implies that $x_1 \mapsto \mu_x(x_1, x_2)$ is decreasing on $[b_1 \vee x, a_1]$. Similarly, for any $x_1 \in [b_1 \vee x, a_1]$, it follows from $x - x_1 \leq x - (b_1 \vee x) \leq 0$ that

$$\frac{\partial}{\partial x_2}\mu_x(x_1, x_2) = \frac{x - x_1}{(x_1 - x_2)^2} \leq 0 \text{ for all } x_2 \in [b_2, a_2].$$

Thus $x_2 \mapsto \mu_x(x_1, x_2)$ is nonincreasing on $[b_2, a_2]$. We can conclude that

$$\begin{aligned} q_x^{\min} = \mu_x(a_1, a_2) &= \min_{x_1 \in [b_1 \vee x, a_1], x_2 \in [b_2, a_2]} \mu_x(x_1, x_2) = \min \bar{\mathcal{Q}}_x, \\ q_x^{\max} = \mu_x(b_1 \vee x, b_2) &= \max_{x_1 \in [b_1 \vee x, a_1], x_2 \in [b_2, a_2]} \mu_x(x_1, x_2) = \max \bar{\mathcal{Q}}_x. \end{aligned}$$

Observe that $q_x^{\min} \leq q_x^{\max}$. Thus (4.84) holds true.

Thirdly, we consider the case when $b_2 \leq x \leq a_2$. From (4.35), we have $0 \in \bar{\mathcal{Q}}_x$ and hence

$$q_x^{\min} = 0 = \min \bar{\mathcal{Q}}_x.$$

Then $q_x^{\min} \leq q_x^{\max}$ because $q_x^{\min} = 0$ and $q_x^{\max} \in [0, 1]$. We will show that

$$q_x^{\max} = \max \bar{\mathcal{Q}}_x$$

by considering the following three situations of the relationship between x and $[b_1, a_1]$.

1. If $b_1 \leq x \leq a_1$, then (4.34) gives $1 \in \bar{\mathcal{Q}}_x$. This means

$$q_x^{\max} = 1 = \max \bar{\mathcal{Q}}_x.$$

2. If $x < b_1$, then for any $x_2 > x$, we have

$$\gamma x_1 + (1 - \gamma)\, x_2 > x \text{ for all } \gamma \in [0, 1], x_1 \in [b_1, a_1].$$

Combining this with the formulation of $\bar{\mathcal{Q}}_x$ in (4.83), it follows that

$$\begin{aligned}\bar{\mathcal{Q}}_x &= \{\gamma \in [0,1] \,|\exists\, x_1 \in [b_1, a_1], x_2 \in [b_2, a_2 \wedge x] : \gamma x_1 + (1-\gamma)\, x_2 = x\} \\ &= \left\{\gamma \in [0,1] \,\middle|\, \exists\, x_1 \in [b_1, a_1], x_2 \in [b_2, a_2 \wedge x] : \gamma = \frac{x - x_2}{x_1 - x_2}\right\},\end{aligned}$$

where $(a_2 \wedge x) < b_1$ which means that there is no overlap between the intervals $[b_1, a_1]$ and $[b_2, a_2 \wedge x]$. Then it follows from

$$\mu_x(x_1, x_2) = \frac{x - x_2}{x_1 - x_2} \in [0,1) \text{ for all } x_1 \in [b_1, a_1], x_2 \in [b_2, a_2 \wedge x]$$

that

$$\bar{\mathcal{Q}}_x = \{\mu_x(x_1, x_2) \,| x_1 \in [b_1, a_1], x_2 \in [b_2, a_2 \wedge x]\}.$$

For any $x_2 \in [b_2, a_2 \wedge x]$, we have from $x_2 - x \leq x_2 - (a_2 \wedge x) \leq 0$ that

$$\frac{\partial}{\partial x_1}\mu_x(x_1, x_2) = \frac{x_2 - x}{(x_1 - x_2)^2} \leq 0 \text{ for all } x_1 \in [b_1, a_1].$$

This implies that $x_1 \mapsto \mu_x(x_1, x_2)$ is nonincreasing on $[b_1, a_1]$. Similarly, for any $x_1 \in [b_1, a_1]$, it follows from $x - x_1 < b_1 - x_1 \leq 0$ that

$$\frac{\partial}{\partial x_2}\mu_x(x_1, x_2) = \frac{x - x_1}{(x_1 - x_2)^2} < 0 \text{ for all } x_2 \in [b_2, a_2 \wedge x].$$

Thus $x_2 \mapsto \mu_x(x_1, x_2)$ is decreasing on $[b_2, a_2 \wedge x]$. Therefore, we can conclude that

$$q_x^{\max} = \mu_x(b_1, b_2) = \max_{x_1 \in [b_1, a_1], x_2 \in [b_2, a_2 \wedge x]} \mu_x(x_1, x_2) = \max \bar{\mathcal{Q}}_x.$$

3. If $x > a_1$, then for any $x_2 < x$, we have

$$\gamma x_1 + (1-\gamma)\, x_2 < x \text{ for all } \gamma \in [0,1], x_1 \in [b_1, a_1].$$

Combining this with the formulation of $\bar{\mathcal{Q}}_x$ in (4.83), it follows that

$$\begin{aligned}\bar{\mathcal{Q}}_x &= \{\gamma \in [0,1] \,|\exists\, x_1 \in [b_1, a_1], x_2 \in [b_2 \vee x, a_2] : \gamma x_1 + (1-\gamma)\, x_2 = x\} \\ &= \left\{\gamma \in [0,1] \,\middle|\, \exists\, x_1 \in [b_1, a_1], x_2 \in [b_2 \vee x, a_2] : \gamma = \frac{x - x_2}{x_1 - x_2}\right\},\end{aligned}$$

where $a_1 < (b_2 \vee x)$ which means that there is no overlap between the intervals $[b_1, a_1]$ and $[b_2 \vee x, a_2]$. Then it follows from

$$\mu_x(x_1, x_2) = \frac{x - x_2}{x_1 - x_2} \in [0,1) \text{ for all } x_1 \in [b_1, a_1], x_2 \in [b_2 \vee x, a_2]$$

that

$$\bar{\mathcal{Q}}_x = \{\mu_x(x_1, x_2)\,|x_1 \in [b_1, a_1], x_2 \in [b_2 \vee x, a_2]\}\,.$$

For any $x_2 \in [b_2 \vee x, a_2]$, we have from $x_2 - x \geq x_2 - (b_2 \vee x) \geq 0$ that

$$\frac{\partial}{\partial x_1}\mu_x(x_1, x_2) = \frac{x_2 - x}{(x_1 - x_2)^2} \geq 0 \text{ for all } x_1 \in [b_1, a_1].$$

This implies that $x_1 \mapsto \mu_x(x_1, x_2)$ is nondecreasing on $[b_1, a_1]$. Similarly, for any $x_1 \in [b_1, a_1]$, it follows from $x - x_1 > a_1 - x_1 \geq 0$ that

$$\frac{\partial}{\partial x_2}\mu_x(x_1, x_2) = \frac{x - x_1}{(x_1 - x_2)^2} > 0 \text{ for all } x_2 \in [b_2 \vee x, a_2].$$

Thus $x_2 \mapsto \mu_x(x_1, x_2)$ is increasing on $[b_2 \vee x, a_2]$. Then we can conclude that

$$q_x^{\max} = \mu_x(a_1, a_2) = \max_{x_1 \in [b_1, a_1], x_2 \in [b_2 \vee x, a_2]} \mu_x(x_1, x_2) = \max \bar{\mathcal{Q}}_x.$$

Therefore (4.84) holds true. Combining (4.84) and $q_x^{\min}, q_x^{\max} \in [0, 1]$, we have $0 \leq q_x^{\min} \leq q_x^{\max} \leq 1$ and $\bar{\mathcal{Q}}_x = \left[q_x^{\min}, q_x^{\max}\right]$, which completes the proof. $\square$

Proof of Lemma 4.20. Consider the cases $\mathrm{C}^{1,1}$ and $\mathrm{C}^{1,2}$. Fix any $\gamma \in \mathcal{Q}_x$. By $\alpha_1 < \alpha_2$ and Proposition 4.15, the value $z_{\gamma,x}$ defined in (4.62) can be written as

$$z_{\gamma,x} = \max \mathcal{Z}_{\gamma,x} = a_1 \wedge \psi_{\gamma,x}^{-1}(b_2).$$

In the case $\mathrm{C}^{1,1}$, it follows from $a_1 < b_2$ that

$$\begin{aligned} \gamma \leq \frac{x - b_2}{a_1 - b_2} \quad &\Longleftrightarrow \quad \gamma(a_1 - b_2) \geq x - b_2 \\ &\Longleftrightarrow \quad a_1 \geq \frac{x - b_2}{\gamma} + b_2 = \frac{x - (1-\gamma)b_2}{\gamma} = \psi_{\gamma,x}^{-1}(b_2) \end{aligned}$$

Observe from (4.67) that $\frac{x - b_2}{a_1 - b_2} = q_x^1$, and hence

$$\gamma \leq q_x^1 \Longleftrightarrow a_1 \geq \psi_{\gamma,x}^{-1}(b_2). \tag{4.85}$$

Similarly, by straightforward calculation, we also have

$$\gamma \geq q_x^1 \Longleftrightarrow a_1 \leq \psi_{\gamma,x}^{-1}(b_2). \tag{4.86}$$

Consider the following two cases. If $\gamma \in \mathcal{Q}_{1,x}^{1,1}$, then $\gamma \leq q_x^1$ which is equivalent to $a_1 \geq \psi_{\gamma,x}^{-1}(b_2)$ by (4.85), and hence $z_{\gamma,x} = \psi_{\gamma,x}^{-1}(b_2)$. If $\gamma \in \mathcal{Q}_{2,x}^{1,1}$, then $\gamma \geq q_x^1$ which is equivalent to $a_1 \leq \psi_{\gamma,x}^{-1}(b_2)$ by (4.86), which means $z_{\gamma,x} = a_1$. The conclusion is that

$$z_{\gamma,x} = \max \mathcal{Z}_{\gamma,x} = \begin{cases} \psi_{\gamma,x}^{-1}(b_2) & \text{if } \gamma \in \mathcal{Q}_{1,x}^{1,1}, \\ a_1 & \text{if } \gamma \in \mathcal{Q}_{2,x}^{1,1}. \end{cases}$$

In the case $\mathrm{C}^{1,2}$, we have $a_1 > b_2$ instead of $a_1 < b_2$ in the case $\mathrm{C}^{1,1}$. In such situation, it follows from straightforward calculation that

$$\gamma \leq q_x^1 \iff a_1 \leq \psi_{\gamma,x}^{-1}(b_2) \tag{4.87}$$

$$\gamma \geq q_x^1 \iff a_1 \geq \psi_{\gamma,x}^{-1}(b_2) \tag{4.88}$$

(cf. (4.85)-(4.86)). Consider the following two cases. If $\gamma \in \mathcal{Q}_{1,x}^{1,2}$, then $\gamma \geq q_x^1$ which is equivalent to $a_1 \geq \psi_{\gamma,x}^{-1}(b_2)$ by (4.88), and hence $z_{\gamma,x} = \psi_{\gamma,x}^{-1}(b_2)$. If $\gamma \in \mathcal{Q}_{2,x}^{1,2}$, then $\gamma \leq q_x^1$ which is equivalent to $a_1 \leq \psi_{\gamma,x}^{-1}(b_2)$ by (4.87), which means $z_{\gamma,x} = a_1$. The conclusion is that

$$z_{\gamma,x} = \max \mathcal{Z}_{\gamma,x} = \begin{cases} \psi_{\gamma,x}^{-1}(b_2) & \text{if } \gamma \in \mathcal{Q}_{1,x}^{1,2}, \\ a_1 & \text{if } \gamma \in \mathcal{Q}_{2,x}^{1,2}. \end{cases}$$

For each $j = 1, 2$, in the situation when $\mathcal{Q}_{1,x}^{1,j} \neq \emptyset$ and $\mathcal{Q}_{2,x}^{1,j} \neq \emptyset$, we have $\mathcal{Q}_{1,x}^{1,j} \cap \mathcal{Q}_{2,x}^{1,j} = \{q_x^1\}$ and moreover

$$z_{\gamma,x} = \psi_{\gamma,x}^{-1}(b_2) = a_1 \text{ for } \gamma = q_x^1$$

(see (4.68)). This completes the proof. □

Proof of Lemma 4.21. Consider the cases $\mathrm{C}^{2,1}$ and $\mathrm{C}^{2,2}$. Fix any $\gamma \in \mathcal{Q}_x$. By $\alpha_1 > \alpha_2$ and Proposition 4.15, the value $z_{\gamma,x}$ defined in (4.62) can be written as

$$z_{\gamma,x} = \min \mathcal{Z}_{\gamma,x} = b_1 \vee \psi_{\gamma,x}^{-1}(a_2).$$

In the case $\mathrm{C}^{2,1}$, it follows from $b_1 < a_2$ that

$$\begin{aligned} \gamma \geq \frac{x - a_2}{b_1 - a_2} \quad &\iff \quad \gamma(b_1 - a_2) \leq x - a_2 \\ &\iff \quad b_1 \leq \frac{x - a_2}{\gamma} + a_2 = \frac{x - (1-\gamma)a_2}{\gamma} = \psi_{\gamma,x}^{-1}(a_2) \end{aligned}$$

Observe from (4.69) that

$$\frac{x - a_2}{b_1 - a_2} = q_x^2,$$

and hence

$$\gamma \geq q_x^2 \iff b_1 \leq \psi_{\gamma,x}^{-1}(a_2). \tag{4.89}$$

Similarly, by straightforward calculation, we also have

$$\gamma \leq q_x^2 \Longleftrightarrow b_1 \geq \psi_{\gamma,x}^{-1}(a_2). \tag{4.90}$$

Consider the following two cases. If $\gamma \in \mathcal{Q}_{1,x}^{2,1}$, then $\gamma \geq q_x^2$ which is equivalent to $b_1 \leq \psi_{\gamma,x}^{-1}(a_2)$ by (4.89), and hence $z_{\gamma,x} = \psi_{\gamma,x}^{-1}(a_2)$. If $\gamma \in \mathcal{Q}_{2,x}^{2,1}$, then $\gamma \leq q_x^2$ which is equivalent to $b_1 \geq \psi_{\gamma,x}^{-1}(a_2)$ by (4.90), which means $z_{\gamma,x} = b_1$. The conclusion is that

$$z_{\gamma,x} = \min \mathcal{Z}_{\gamma,x} = \begin{cases} \psi_{\gamma,x}^{-1}(a_2) & \text{if } \gamma \in \mathcal{Q}_{1,x}^{2,1}, \\ b_1 & \text{if } \gamma \in \mathcal{Q}_{2,x}^{2,1}. \end{cases}$$

In the case $\mathrm{C}^{2,2}$, we have $b_1 > a_2$ instead of $b_1 < a_2$ in the case $\mathrm{C}^{2,1}$. In such situation, it follows from straightforward calculation that

$$\gamma \geq q_x^2 \Longleftrightarrow b_1 \geq \psi_{\gamma,x}^{-1}(a_2) \tag{4.91}$$

$$\gamma \leq q_x^2 \Longleftrightarrow b_1 \leq \psi_{\gamma,x}^{-1}(a_2) \tag{4.92}$$

(cf. (4.89)-(4.90)). Consider the following two cases. If $\gamma \in \mathcal{Q}_{1,x}^{2,2}$, then $\gamma \leq q_x^2$ which is equivalent to $b_1 \leq \psi_{\gamma,x}^{-1}(a_2)$ by (4.92), and hence $z_{\gamma,x} = \psi_{\gamma,x}^{-1}(a_2)$. If $\gamma \in \mathcal{Q}_{2,x}^{2,2}$, then $\gamma \geq q_x^2$ which is equivalent to $b_1 \geq \psi_{\gamma,x}^{-1}(a_2)$ by (4.91), which means $z_{\gamma,x} = b_1$. The conclusion is that

$$z_{\gamma,x} = \min \mathcal{Z}_{\gamma,x} = \begin{cases} \psi_{\gamma,x}^{-1}(a_2) & \text{if } \gamma \in \mathcal{Q}_{1,x}^{2,2}, \\ b_1 & \text{if } \gamma \in \mathcal{Q}_{2,x}^{2,2}. \end{cases}$$

For each $j = 1, 2$, in the situation when $\mathcal{Q}_{1,x}^{2,j} \neq \emptyset$ and $\mathcal{Q}_{2,x}^{2,j} \neq \emptyset$, we have $\mathcal{Q}_{1,x}^{2,j} \cap \mathcal{Q}_{2,x}^{2,j} = \{q_x^2\}$ and moreover

$$z_{\gamma,x} = \psi_{\gamma,x}^{-1}(a_2) = b_1 \text{ for } \gamma = q_x^2.$$

(see (4.70)). This completes the proof. □

4.3.2 Piecewise linear case

Section 4.3.1 above provides a method to construct a solution to the problem (4.31) with $x \in \operatorname{dom} f$, where f_1 and f_2 in (4.31) are affine on their effective domains. In this section, we still focus on the problem (4.31), but f_1 and f_2 are allowed to be piecewise linear on their effective domains.

For each $i = 1, 2$, let $f_i : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a convex function that is continuous and piecewise linear on $\operatorname{dom} f_i$, where $\operatorname{dom} f_i$ is assumed to be a closed interval. Moreover, the function f_i is affine on each of the following $n_i \geq 1$ closed intervals

$$D_i^1 = \left[y_i^1, y_i^2\right], D_i^2 = \left[y_i^2, y_i^3\right], \dots, D_i^{n_i} = \left[y_i^{n_i}, y_i^{n_i+1}\right] \subseteq \operatorname{dom} f_i,$$

where

$$\min(\operatorname{dom} f_i) = y_i^1 \leq y_i^2 \leq \cdots \leq y_i^{n_i+1} = \max(\operatorname{dom} f_i).$$

Notice that

$$\bigcup_{k=1}^{n_i} D_i^k = \operatorname{dom} f_i.$$

For all $x \in \mathbb{R}$, let

$$f(x) = \inf \left\{ \sum_{i=1,2} \left(\lambda_i f_i(x_i) + g_i^1(\lambda_i)\right) \middle| \right.$$
$$\left. \lambda_i \in [0,1], x_i \in \operatorname{dom} f_i \, \forall i = 1,2, \sum_{i=1,2} \lambda_i = 1, \sum_{i=1,2} \lambda_i x_i = x \right\}. \quad (4.93)$$

It follows from Theorem 4.3 that f is an $\mathbb{R} \cup \{\infty\}$-valued convex function on $\mathbb{R}$, and moreover

$$\operatorname{dom} f = \operatorname{co}\left(\operatorname{dom} f_1 \cup \operatorname{dom} f_2\right).$$

This implies that $f(x) = \infty$ for all $x \notin \operatorname{co}\left(\operatorname{dom} f_1 \cup \operatorname{dom} f_2\right)$. For the remainder of this section, let

$$x^* \in \operatorname{co}\left(\operatorname{dom} f_1 \cup \operatorname{dom} f_2\right);$$

the quantity x^* is fixed throughout this section. Then $f(x^*) \in \mathbb{R}$. From Theorem 4.13, there exists a solution to the minimisation problem in (4.93) with $x = x^*$, in other words, there exists $(\lambda_1, x_1, \lambda_2, x_2)$ such that the constraints in (4.93) are satisfied for $x = x^*$ and

$$\sum_{i=1,2} \left(\lambda_i f_i(x_i) + g_i^1(\lambda_i)\right) = f(x^*).$$

In this remainder of section, we will provide a method to find a solution to the problem in (4.93) with $x = x^*$.

For all $k_1 = 1, \dots, n_1$ and $k_2 = 1, \dots, n_2$, we define $f_1^{k_1}, f_2^{k_2} : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ as

$$f_1^{k_1} = \begin{cases} f_1 & \text{on } D_1^{k_1}, \\ \infty & \text{on } \mathbb{R} \backslash D_1^{k_1}, \end{cases} \qquad f_2^{k_2} = \begin{cases} f_2 & \text{on } D_2^{k_2}, \\ \infty & \text{on } \mathbb{R} \backslash D_2^{k_2}. \end{cases}$$

Observe that $f_1^{k_1} = f_1$ is an affine function on $\operatorname{dom} f_1^{k_1} = D_1^{k_1}$, and $f_2^{k_2} = f_2$

is an affine function on $\operatorname{dom} f_2^{k_2} = D_2^{k_2}$. Moreover, we define

$$f^{k_1,k_2}(x^*) := \inf \left\{ \sum_{i=1,2} \left(\lambda_i f_i^{k_i}(x_i) + g_i^1(\lambda_i) \right) \middle| \vphantom{\sum} \right.$$
$$\left. \lambda_i \in [0,1], x_i \in D_i^{k_i}\ \forall i = 1,2,\ \sum_{i=1,2} \lambda_i = 1,\ \sum_{i=1,2} \lambda_i x_i = x^* \right\}; \quad (4.94)$$

cf. (4.93). Clearly, we have $f^{k_1,k_2}(x^*) \in \mathbb{R} \cup \{\infty\}$, and moreover $f^{k_1,k_2}(x^*) \in \mathbb{R}$ if and only if $x^* \in \operatorname{co}\left(D_1^{k_1} \cup D_2^{k_2}\right)$. In the situation when $f^{k_1,k_2}(x^*) \in \mathbb{R}$, we can use the method in Section 4.3.1 to find $(\lambda_1, x_1, \lambda_2, x_2)$ such that the constraints in (4.94) are satisfied and

$$\sum_{i=1,2} \left(\lambda_i f_i^{k_i}(x_i) + g_i^1(\lambda_i) \right) = f^{k_1,k_2}(x^*);$$

see (4.44) and Theorems 4.18, 4.19, and 4.27 for the main results in Section 4.3.1. We call such $(\lambda_1, x_1, \lambda_2, x_2)$ a solution to the problem in (4.94).

In the remainder of this section, let $k_1' = \{1, \ldots, n_1\}$ and $k_2' \in \{1, \ldots, n_2\}$ be the integers such that

$$f^{k_1',k_2'}(x^*) = \min \left\{ f^{k_1,k_2}(x^*) \middle| k_1 = 1, \ldots, n_1,\ k_2 = 1, \ldots, n_2 \right\};$$

the integers k_1' and k_2' may depend on the choice of x^*. Such k_1' and k_2' always exist (but may not be unique) because n_1 and n_2 are finite. The following result shows that the values of $f(x^*)$ and $f^{k_1',k_2'}(x^*)$ are the same.

Theorem 4.28. *We have*

$$f(x^*) = f^{k_1',k_2'}(x^*).$$

Proof. Fix any $k_1 = 1, \ldots, n_1$ and $k_2 = 1, \ldots, n_2$. For any $(\lambda_1, x_1, \lambda_2, x_2)$ such that the constraints in (4.94) are satisfied, it follows from $x_i \in D_i^{k_i} \subseteq \operatorname{dom} f_i$ for each $i = 1, 2$ that the constraints in (4.93) with $x = x^*$ are also satisfied. Then the definition of $f(x^*)$ implies

$$f(x^*) \leq \sum_{i=1,2} \left(\lambda_i f_i(x_i) + g_i^1(\lambda_i) \right).$$

Combining this with $f_i(x_i) = f_i^{k_1}(x_i)$ for each $i = 1, 2$, it follows that

$$f(x^*) \leq \sum_{i=1,2} \left(\lambda_i f_i^{k_i}(x_i) + g_i^1(\lambda_i) \right).$$

Taking infimum on both sides over all $(\lambda_1, x_1, \lambda_2, x_2)$ that satisfies the constraints in (4.94), it follows that $f(x^*) \leq f^{k_1,k_2}(x)$. In particular, by letting $k_1 = k_1'$ and $k_2 = k_2'$, we have

$$f(x^*) \leq f^{k_1',k_2'}(x^*).$$

We are going to show that the opposite inequality also holds true. By Theorem 4.13, there exists a solution $(\lambda_1, x_1, \lambda_2, x_2)$ to the problem (4.93) with $x = x^*$, which means

$$f(x^*) = \sum_{i=1,2} \Big(\lambda_i f_i(x_i) + g_i^1(\lambda_i)\Big).$$

Notice that

$$x_1 \in \operatorname{dom} f_1 = \bigcup_{k=1}^{n_1} D_1^k, \qquad\qquad x_2 \in \operatorname{dom} f_2 = \bigcup_{k=1}^{n_2} D_2^k.$$

Then $x_1 \in D_1^{k_1}$ and $x_2 \in D_2^{k_2}$ for some $k_1 = 1, \dots, n_1$ and $k_2 = 1, \dots, n_2$, and hence

$$f_1(x_1) = f_1^{k_1}(x_1), \qquad\qquad f_2(x_2) = f_2^{k_2}(x_2).$$

Therefore

$$f(x^*) = \sum_{i=1,2} \Big(\lambda_i f_i^{k_i}(x_i) + g_i^1(\lambda_i)\Big) \geq f^{k_1,k_2}(x^*) \geq f^{k_1',k_2'}(x^*),$$

where the first inequality follows from the definition of $f^{k_1,k_2}(x^*)$ in (4.94) and the fact that $(\lambda_1, x_1, \lambda_2, x_2)$ satisfies the constraints in (4.94). Therefore, the result follows. $\square$

By Theorem 4.28 and $f(x^*) \in \mathbb{R}$, we must have $f^{k_1',k_2'}(x^*) \in \mathbb{R}$. This means that there exists a solution to the problem in (4.94) with $(k_1, k_2) = (k_1', k_2')$; see the comments following (4.94). The following result shows the problem in (4.93) with $x = x^*$ can be solved by solving the problem in (4.94) with $(k_1, k_2) = (k_1', k_2')$.

Theorem 4.29. *A solution to the problem in* (4.94) *with* $(k_1, k_2) = (k_1', k_2')$ *is also a solution to the problem in* (4.93) *with* $x = x^*$*.*

Proof. Firstly, let $(\lambda_1, x_1, \lambda_2, x_2)$ be a solution to the problem in (4.94) with $(k_1, k_2) = (k_1', k_2')$. Then $(\lambda_1, x_1, \lambda_2, x_2)$ satisfies the constraints in (4.94) with $(k_1, k_2) = (k_1', k_2')$. Combining this with the fact that $x_i \in D_i^{k_i'} \subseteq \operatorname{dom} f_i$ for each $i = 1, 2$, it follows that $(\lambda_1, x_1, \lambda_2, x_2)$ also satisfies the constraints in

(4.93) with $x = x^*$. Since $f_i(x_i) = f_i^{k_i'}(x_i)$ for each $i = 1, 2$, we have

$$\sum_{i=1,2} \left(\lambda_i f_i(x_i) + g_i^1(\lambda_i)\right) = \sum_{i=1,2} \left(\lambda_i f_i^{k_i'}(x_i) + g_i^1(\lambda_i)\right) = f^{k_1',k_2'}(x^*) = f(x^*)$$

(Theorem 4.28). Therefore $(\lambda_1, x_1, \lambda_2, x_2)$ is a solution to the problem in (4.93) with $x = x^*$. $\square$

Chapter 5

Optimisation with exponential regret

In Example 3.26, we specified a sequence of regret functions to measure the investor's regret upon cash injection. Moreover, we provide an explicit formula for the objective function of the dual optimisation problem (3.35). In this chapter, based on the market model introduced in Chapter 2 with the robust no-arbitrage condition being assumed, we will study the problem (3.35) in detail with the regret functions used in Example 3.26. This chapter is organised as follows.

Section 5.1 will first briefly review the dual optimisation problem (3.35). Then the problem will be written as two nested optimisation problems. The first one, which is problem (5.7), will be studied in Section 5.2 in detail, and the other one can be solved explicitly; see Proposition 5.4. Then Theorem 5.5 provides a formula to compute the minimal regret of the investor with any given liabilities. In addition, Theorem 5.6 provides a method to construct an optimal injection strategy for the problem (3.19) via a solution to the problem (3.35). After that, we will provide a formula in Theorem 5.7 to compute the regret indifference prices introduced in (3.51) and (3.52). In Example 5.10, we will consider a one-step toy model, and we will solve the problem (5.7) explicitly. Moreover, the formulae from Theorem 5.7 will be applied to compute the indifference prices of a European call option.

In Section 5.2, we will first make an assumption that the bid-ask stock prices satisfy (5.26). Then Section 5.2.1 introduces the notion of transition probabilities and provides a number of technical results that will be used in later sections. In Section 5.2.2, we will provide Algorithm 5.17 to construct a sequence of random functions $(J_t)_{t=0}^T$. Based on $(J_t)_{t=0}^T$, a pair $(\hat{\mathbb{Q}}, \hat{S})$ can be constructed from Algorithm 5.19. Theorem 5.20 will show that $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$

and it is a solution to the problem (5.7).

Since $(J_t)_{t=0}^T$ constructed in Algorithm 5.17 is difficult to calculate, Section 5.3 will introduce a piecewise linear approximation $(\widetilde{J}_t)_{t=0}^T$ to approximate $(J_t)_{t=0}^T$. For each $t = 0, \dots, T$, we always have $J_t \leq \widetilde{J}_t$, and hence $\widetilde{J}_t$ is an upper bound of J_t. At the end of this section, Theorem 5.25 will provide the relevant convergent result.

In Section 5.4, we will provide a method to compute the approximation error of approximating $(J_t)_{t=0}^T$ by using $(\widetilde{J}_t)_{t=0}^T$. To achieve this, we will construct a sequence of random functions $(\check{J}_t)_{t=0}^T$ such that $\check{J}_t \leq J_t$ for all $t = 0, \dots, T$. Then $\widetilde{J}_t - \check{J}_t$ is an upper bound of the approximation error $\widetilde{J}_t - J_t$. We can calculate this upper bound by using the results established in Section 4.3.

Finally, we will introduce a binary market model in Section 5.5. In this model, Section 5.5.1 will provide numerical examples to compute the error of approximating $(J_t)_{t=0}^T$ by $(\widetilde{J}_t)_{t=0}^T$. Then Section 5.5.2 will numerically compute the solution to the problem (3.19). Moreover, in Section 5.5.3, we will provide numerical examples to compute the indifference prices.

5.1 Minimal regret, hedging and pricing

In this section, we will consider the dual problem (3.35) under the exponential regret functions introduced in Example 3.26. Firstly, we will write (3.35) as two nested optimisation problems. The first problem appears in (5.7), and we will discuss how to solve this problem in Section 5.2. The second problem appears in (5.10), and it can be solved explicitly with the optimal value of the first problem; see Proposition 5.4. After the study of (3.35), by applying the strong duality established in Theorem 3.31, we will derive a formula for computing the minimal regret defined in (3.9); see Theorem 5.5. Then Theorem 5.6 shows that an optimal injection to the problem (3.19) can be constructed via a solution to (3.35). At the end of this section, we will provide formulae in Theorem 5.7 to calculate the indifference prices defined in (3.51)-(3.52). By applying these formulae in Example 5.10, we will compute the indifference prices of a European call option in a one-step toy model.

We shall specify the regret functions $(v_t)_{t=0}^T$ used in optimisation problem (3.8) as follows; the construction of $(v_t)_{t=0}^T$ follows from Example 3.26. First of all, let

$$\mathcal{I} := \{t_1, \dots, t_n, T\} \subseteq \{0, \dots, T\}$$

be a collection of time steps. Moreover, let $(\alpha_t)_{t \in \mathcal{I}}$ be a sequence of positive numbers (i.e. $\alpha_t \in (0, \infty)$ for all $t \in \mathcal{I}$). Notice that $t \mapsto \alpha_t$ is deterministic.

Then, for all $t \in \{0, \dots, T\}$ and $\omega \in \Omega$, we define

$$v_t^\omega(x) := \begin{cases} e^{\alpha_t x} - 1 & \text{if } t \in \mathcal{I}, \\ \delta_{(-\infty,0]}(x) & \text{if } t \notin \mathcal{I}. \end{cases} \tag{5.1}$$

At every time step $t \in \mathcal{I}$, the investor's regret upon injecting x units of cash is always measured by the real number $e^{\alpha_t x} - 1$. At each time step $t \in \{0, \dots, T\} \backslash \mathcal{I}$, the investor's regret upon any positive injection is infinity, but his regret is zero with any withdrawals (i.e. negative injections). This completes the construction of $(v_t)_{t=0}^T$. Consider the following two special examples of $\mathcal{I}$:

$$\mathcal{I}^R := \{0, \dots, T\}, \tag{5.2}$$

$$\mathcal{I}^U := \{T\}. \tag{5.3}$$

Observe that when $\mathcal{I} = \mathcal{I}^R$ the investor is allowed to inject arbitrary amount of cash at every time step $t = 0, \dots, T$. However, when $\mathcal{I} = \mathcal{I}^U$, the investor is only allowed to have positive injections at time T. Combining this with Example 3.13, under $(v_t)_{t=0}^T$ specified in (5.1), the optimisation problem (3.8) is closely connected to utility maximisation problems. Both $\mathcal{I}^R$ and $\mathcal{I}^U$ will be used frequently in Section 5.5.

Remark 5.1. The construction of $(\alpha_t)_{t\in\mathcal{I}}$ depends on one's modelling purpose. At time $t \in \mathcal{I}$, the value α_t is used to model the investor's risk aversion on cash injection. For example, one can take $t \mapsto \alpha_t$ to be constant in order to model constant risk aversion over time. Similarly, one can also model increasing (resp. decreasing) risk aversion over time by setting $t \mapsto \alpha_t$ to be increasing (resp. decreasing). In Examples 5.44 and 5.47, we will provide numerical examples of optimal injections to the problem (3.19) for various different $(\alpha_t)_{t\in\mathcal{I}}$.

Fix any $u = (u_t)_{t=0}^T \in \mathcal{N}^2$. The function

$$(\lambda, (\mathbb{Q}, S)) \mapsto \inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) \text{ on } [0, \infty) \in \bar{\mathcal{P}}$$

is the objective function of the problem (3.35). The following result provides a representation for this function, and this result will be used to tackle (3.35). Let

$$|\mathcal{I}| := \sum_{t\in\mathcal{I}} 1$$

be the number of time steps in $\mathcal{I}$.

Proposition 5.2. *For any* $(\lambda, (\mathbb{Q}, S)) \in [0, \infty) \in \bar{\mathcal{P}}$, *we have*

$$\inf_{x \in \mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) = \lambda \left(\mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \sum_{t=0}^{T} u_t \right] - \sum_{t \in \mathcal{I}} \frac{1}{\alpha_t} \mathbb{E} \left[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}} \right] \right) - \sum_{t \in \mathcal{I}} \left(\frac{\lambda}{\alpha_t} \ln \frac{\lambda}{\alpha_t} - \frac{\lambda}{\alpha_t} \right) - |\mathcal{I}| . \quad (5.4)$$

Proof. We have from Example 3.26 that

$$\inf_{x \in \mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) = \lambda \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot \textstyle\sum_{t=0}^{T} u_t \right] - \sum_{t \in \mathcal{I}} \mathbb{E} \left[\frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} \ln \frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} - \frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} \right] - |\mathcal{I}| .$$

Fix any $t \in \mathcal{I}$. Observe that

$$\mathbb{E} \left[\frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} \ln \frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} - \frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} \right] = \mathbb{E} \left[\frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} \ln \frac{\lambda}{\alpha_t} \right] + \mathbb{E} \left[\frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} \ln \Lambda_t^{\mathbb{Q}} \right] - \mathbb{E} \left[\frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} \right].$$

Moreover, since α_t is deterministic and $\mathbb{E}[\Lambda_t^{\mathbb{Q}}] = \mathbb{E}_{\mathbb{Q}}[1] = 1$, it follows that

$$\begin{aligned} \mathbb{E} \left[\frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} \ln \frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} - \frac{\lambda \Lambda_t^{\mathbb{Q}}}{\alpha_t} \right] &= \frac{\lambda}{\alpha_t} \ln \frac{\lambda}{\alpha_t} \mathbb{E} \left[\Lambda_t^{\mathbb{Q}} \right] + \frac{\lambda}{\alpha_t} \mathbb{E} \left[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}} \right] - \frac{\lambda}{\alpha_t} \mathbb{E} \left[\Lambda_t^{\mathbb{Q}} \right] \\ &= \frac{\lambda}{\alpha_t} \ln \frac{\lambda}{\alpha_t} + \frac{\lambda}{\alpha_t} \mathbb{E} \left[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}} \right] - \frac{\lambda}{\alpha_t}. \end{aligned}$$

Then (5.4) follows. $\square$

Our next objective is to show that the problem (3.35) can be viewed as two nested optimisation problems.

Firstly, fix any $\lambda \in [0, \infty)$, and consider the following optimisation problem

$$\text{maximise } \inf_{x \in \mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) \text{ over } (\mathbb{Q}, S) \in \bar{\mathcal{P}}. \quad (5.5)$$

Fix any $X \in \mathcal{L}_T^2$. For convenience, for any $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$, let

$$H_{\mathcal{I}} \left((\mathbb{Q}, S); X \right) := \sum_{t \in \mathcal{I}} \frac{1}{\alpha_t} \mathbb{E} \left[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}} \right] + \mathbb{E}_{\mathbb{Q}} \left[(1, S_T) \cdot X \right]. \quad (5.6)$$

Moreover, we define

$$K_{\mathcal{I}} (X) := \inf \left\{ H_{\mathcal{I}} \left((\mathbb{Q}, S); X \right) \middle| (\mathbb{Q}, S) \in \bar{\mathcal{P}} \right\}. \quad (5.7)$$

Notice that $K_{\mathcal{I}} (X)$ is finite because the values of $x \mapsto x \ln x$ are finite and bounded from below on $[0, \infty)$. In Section 5.2, under the condition (5.26), we will provide a method to construct $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$ such that

$$H_{\mathcal{I}}\left((\hat{\mathbb{Q}}, \hat{S}); X\right) = K_{\mathcal{I}}(X);$$

see Theorem 5.20. Observe that

$$K_{\mathcal{I}}(X + (\delta, 0)) = K_{\mathcal{I}}(X) + \delta \text{ for all } \delta \in \mathbb{R}. \tag{5.8}$$

Moreover, combining Proposition 5.2 and (5.6)-(5.7), the optimal value of the problem (5.5) can be written as

$$\begin{aligned}\sup_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) = -\lambda K_{\mathcal{I}}\left(-\textstyle\sum_{t=0}^{T} u_t\right) \\ - \sum_{t\in\mathcal{I}} \left(\frac{\lambda}{\alpha_t} \ln \frac{\lambda}{\alpha_t} - \frac{\lambda}{\alpha_t}\right) - |\mathcal{I}|.\end{aligned} \tag{5.9}$$

Thus, in order to solve the problem (5.5), we only need to solve the minimisation problem (5.7) for $X = -\sum_{t=0}^{T} u_t$. The lemma below shows that $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$ as long as $(\hat{\mathbb{Q}}, \hat{S})$ is a solution to (5.7). This result will be used in the proof of Theorem 5.20. The proof of this lemma will be provided at the end of this section.

Lemma 5.3. *If* $(\hat{\mathbb{Q}}, \hat{S}) \in \bar{\mathcal{P}}$ *solves* (5.7)*, then* $\hat{\mathbb{Q}}(\omega) > 0$ *for all* $\omega \in \Omega$*, in other words, we have* $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$*.*

Secondly, consider the following optimisation problem

$$\text{maximise} \sup_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u(x, \lambda, (\mathbb{Q}, S)) \text{ over } \lambda \in [0, \infty). \tag{5.10}$$

The proposition below shows that this optimisation problem can be solved explicitly, and moreover the optimal solution is unique and non-zero. For all $w = (w_t)_{t=0}^{T} \in \mathcal{N}^2$, we define

$$\hat{\lambda}(w) := \exp\left[\frac{1}{\sum_{t\in\mathcal{I}} \frac{1}{\alpha_t}} \left(\sum_{t\in\mathcal{I}} \frac{\ln \alpha_t}{\alpha_t} - K_{\mathcal{I}}\left(-\sum_{t=0}^{T} w_t\right)\right)\right] \in (0, \infty). \tag{5.11}$$

Observe that $\hat{\lambda}(w)$ depends only on $\sum_{t=0}^{T} w_t$, in other words, we have for any $w' = (w'_t)_{t=0}^{T} \in \mathcal{N}^2$ such that $\sum_{t=0}^{T} w'_t = \sum_{t=0}^{T} w_t$ that $\hat{\lambda}(w') = \hat{\lambda}(w)$. Moreover, by straight forward calculation, it follows that

$$\ln\left(\hat{\lambda}(u)\right) \sum_{t\in\mathcal{I}} \frac{1}{\alpha_t} = \sum_{t\in\mathcal{I}} \frac{\ln \alpha_t}{\alpha_t} - K_{\mathcal{I}}\left(-\sum_{t=0}^{T} u_t\right). \tag{5.12}$$

This will be used in the proofs of Proposition 5.4 and Theorem 5.5 below.

Proposition 5.4. *The quantity* $\hat{\lambda}(u)$ *is the unique value in* $[0,\infty)$ *such that*

$$\sup_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u\left(x,\hat{\lambda}(u),(\mathbb{Q},S)\right) = \sup_{\lambda\geq 0}\left\{\sup_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S))\right\}.$$

This means that $\hat{\lambda}(u)$ *is the unique solution to the problem* (5.10).

Proof. For convenience, let

$$f(\lambda) := \sup_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \inf_{x\in\mathcal{N}} L_u(x,\lambda,(\mathbb{Q},S)) \text{ for all } \lambda\in[0,\infty).$$

From (5.9), the function f is continuous on $[0,\infty)$. Moreover, for any $\lambda>0$, by straightforward calculation, the derivatives $f'(\lambda)$ and $f''(\lambda)$ can be presented as

$$f'(\lambda) = -K_{\mathcal{I}}\left(-\sum_{t=0}^{T} u_t\right) - \ln(\lambda)\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t} + \sum_{t\in\mathcal{I}}\frac{\ln\alpha_t}{\alpha_t}$$

and

$$f''(\lambda) = -\frac{1}{\lambda}\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t} < 0.$$

Thus f' is decreasing on $(0,\infty)$. It follows from (5.12) that

$$f'\left(\hat{\lambda}(u)\right) = -K_{\mathcal{I}}\left(-\sum_{t=0}^{T} u_t\right) - \sum_{t\in\mathcal{I}}\frac{\ln\alpha_t}{\alpha_t} + K_{\mathcal{I}}\left(-\sum_{t=0}^{T} u_t\right) + \sum_{t\in\mathcal{I}}\frac{\ln\alpha_t}{\alpha_t} = 0.$$

Combining this with the fact that f' is decreasing on $(0,\infty)$, we have $f'>0$ on $(0,\hat{\lambda}(u))$ and $f'<0$ on $(\hat{\lambda}(u),\infty)$. This implies that f is increasing on $[0,\hat{\lambda}(u)]$ and decreasing on $[\hat{\lambda}(u),\infty)$. Thus, the result follows. ☐

From the strong duality established in Theorem 3.31, the minimal regret $V(u)$ defined in (3.9) is equal to the optimal value of the dual optimisation problem (3.35). This enable us to present $V(u)$ as follows.

Theorem 5.5. *Under the assumption that the robust no-arbitrage condition holds true, the minimal regret* $V(u)$ *can be written as*

$$V(u) = \hat{\lambda}(u)\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t} - |\mathcal{I}|.$$

Proof. Combining Theorem 3.31, Proposition 5.4 and (5.9), we have

$$V(u) = -\hat{\lambda}(u)\,K_{\mathcal{I}}\left(-\sum_{t=0}^{T} u_t\right) - \sum_{t\in\mathcal{I}}\left(\frac{\hat{\lambda}(u)}{\alpha_t}\ln\frac{\hat{\lambda}(u)}{\alpha_t} - \frac{\hat{\lambda}(u)}{\alpha_t}\right) - |\mathcal{I}|. \quad (5.13)$$

Observe that

$$\begin{aligned}\frac{\hat{\lambda}(u)}{\alpha_t}\ln\frac{\hat{\lambda}(u)}{\alpha_t}-\frac{\hat{\lambda}(u)}{\alpha_t} &= \hat{\lambda}(u)\left(\frac{1}{\alpha_t}\ln\frac{\hat{\lambda}(u)}{\alpha_t}-\frac{1}{\alpha_t}\right)\\ &= \hat{\lambda}(u)\left(\frac{1}{\alpha_t}\ln\hat{\lambda}(u)-\frac{1}{\alpha_t}\ln\alpha_t-\frac{1}{\alpha_t}\right)\\ &= \hat{\lambda}(u)\left(\ln\left(\hat{\lambda}(u)\right)\frac{1}{\alpha_t}-\frac{\ln\alpha_t}{\alpha_t}-\frac{1}{\alpha_t}\right).\end{aligned}$$

Then

$$\sum_{t\in\mathcal{I}}\left(\frac{\hat{\lambda}(u)}{\alpha_t}\ln\frac{\hat{\lambda}(u)}{\alpha_t}-\frac{\hat{\lambda}(u)}{\alpha_t}\right)=\hat{\lambda}(u)\left(\ln\left(\hat{\lambda}(u)\right)\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t}-\sum_{t\in\mathcal{I}}\frac{\ln\alpha_t}{\alpha_t}-\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t}\right).$$

Moreover, it follows from (5.12) that

$$\sum_{t\in\mathcal{I}}\left(\frac{\hat{\lambda}(u)}{\alpha_t}\ln\frac{\hat{\lambda}(u)}{\alpha_t}-\frac{\hat{\lambda}(u)}{\alpha_t}\right)=\hat{\lambda}(u)\left(-K_{\mathcal{I}}\left(-\sum_{t=0}^{T}u_t\right)-\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t}\right).$$

Combining this with (5.13), the result follows. □

Suppose that $(\hat{\mathbb{Q}},\hat{S})\in\mathcal{P}$ solves (5.7). Then the theorem below gives a method for computing the solution to (3.19). In Theorem 5.20 below, we will provide an algorithm to construct such $(\hat{\mathbb{Q}},\hat{S})$ under the condition (5.26).

Theorem 5.6. *Under the assumption that the robust no-arbitrage condition holds true, if $(\hat{\mathbb{Q}},\hat{S})\in\mathcal{P}$ solves* (5.7) *with $X=-\sum_{t=0}^{T}u_t$, then the unique solution $(\hat{x}_t)_{t=0}^{T}\in\mathcal{N}$ to the problem* (3.19) *with the regret functions defined in* (5.1) *can be constructed as follows:*

$$\hat{x}_t:=\begin{cases}\frac{1}{\alpha_t}\ln\frac{\hat{\lambda}(u)\Lambda_t^{\hat{\mathbb{Q}}}}{\alpha_t} & \text{if } t\in\mathcal{I},\\ 0 & \text{if } t\in\{0,\dots,T\}\setminus\mathcal{I};\end{cases}$$

here $\hat{\lambda}(u)\Lambda_t^{\hat{\mathbb{Q}}}>0$ for all $t\in\mathcal{I}$ because $\hat{\mathbb{Q}}\sim\mathbb{P}$.

Proof. Combining Proposition 5.2 and (5.6), it follows that

$$\begin{aligned}\inf_{x\in\mathcal{N}}L_u(x,\hat{\lambda}(u),(\hat{\mathbb{Q}},\hat{S})) &= -\hat{\lambda}(u)H_{\mathcal{I}}\left((\hat{\mathbb{Q}},\hat{S});-\textstyle\sum_{t=0}^{T}u_t\right)\\ &\quad-\sum_{t\in\mathcal{I}}\left(\frac{\hat{\lambda}(u)}{\alpha_t}\ln\frac{\hat{\lambda}(u)}{\alpha_t}-\frac{\hat{\lambda}(u)}{\alpha_t}\right)-|\mathcal{I}|.\end{aligned}$$

Since $(\hat{\mathbb{Q}},\hat{S})\in\mathcal{P}$ solves (5.7) with $X=-\sum_{t=0}^{T}u_t$, we have

$$H_{\mathcal{I}}\left((\hat{\mathbb{Q}},\hat{S});-\textstyle\sum_{t=0}^{T}u_t\right)=K_{\mathcal{I}}\left(-\textstyle\sum_{t=0}^{T}u_t\right).$$

This means

$$\begin{aligned}\inf_{x\in\mathcal{N}} L_u(x,\hat{\lambda}\,(u)\,,(\hat{\mathbb{Q}},\hat{S})) = -\hat{\lambda}\,(u)\,K_{\mathcal{I}}\left(-\textstyle\sum_{t=0}^{T} u_t\right)&\\ -\sum_{t\in\mathcal{I}}\left(\frac{\hat{\lambda}\,(u)}{\alpha_t}\ln\frac{\hat{\lambda}\,(u)}{\alpha_t}-\frac{\hat{\lambda}\,(u)}{\alpha_t}\right)-|\mathcal{I}|\,.&\end{aligned}$$

Then it follows from (5.9) and Proposition 5.4 that

$$\begin{aligned}\inf_{x\in\mathcal{N}} L_u(x,\hat{\lambda}\,(u)\,,(\hat{\mathbb{Q}},\hat{S})) &= \sup_{(\mathbb{Q},S)\in\bar{\mathcal{P}}}\inf_{x\in\mathcal{N}} L_u(x,\hat{\lambda}\,(u)\,,(\mathbb{Q},S))\\ &= \sup_{\lambda\geq 0,(\mathbb{Q},S)\in\bar{\mathcal{P}}}\inf_{x\in\mathcal{N}} L_u\left(x,\lambda,(\mathbb{Q},S)\right).\end{aligned}$$

This means that $(\hat{\lambda}\,(u)\,,(\hat{\mathbb{Q}},\hat{S}))$ is a solution to (3.35) with the regret functions defined in (5.1). Then the result follows from Example 3.35. $\square$

In Theorem 5.5, we provided a presentation of the minimal regret. This presentation will be used to derive the formulae in Theorem 5.7 below for the regret indifference prices defined in (3.51)-(3.52).

Theorem 5.7. *Under the assumption that the robust no-arbitrage condition holds true. Then we have for any $c,\bar{c}\in\mathcal{N}^2$ that*

$$\begin{aligned}\pi_{\mathrm{F}}^{ai}\left(c;\bar{c}\right) &= K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}\bar{c}_t\right)-K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}(\bar{c}_t-c_t)\right)\\ \pi_{\mathrm{F}}^{bi}\left(c;\bar{c}\right) &= K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}(\bar{c}_t+c_t)\right)-K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}\bar{c}_t\right).\end{aligned}$$

Proof. Let

$$\delta := K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}\bar{c}_t\right)-K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}(\bar{c}_t-c_t)\right). \tag{5.14}$$

Observe from (5.11) that

$$\hat{\lambda}\left(c-\delta\mathbb{1}-\bar{c}\right)=\exp\left[\frac{1}{\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t}}\left(\sum_{t\in\mathcal{I}}\frac{\ln\alpha_t}{\alpha_t}-K_{\mathcal{I}}\left((\delta,0)-\sum_{t=0}^{T}(c_t-\bar{c}_t)\right)\right)\right].$$

Moreover, combining (5.8) and (5.14), it follows that

$$\begin{aligned}K_{\mathcal{I}}\left((\delta,0)-\textstyle\sum_{t=0}^{T}(c_t-\bar{c}_t)\right) &= K_{\mathcal{I}}\left(-\textstyle\sum_{t=0}^{T}(c_t-\bar{c}_t)\right)+\delta\\ &= K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}\bar{c}_t\right).\end{aligned}$$

Thus, we have

$$\hat{\lambda}\left(c-\delta\mathbb{1}-\bar{c}\right)=\exp\left[\frac{1}{\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t}}\left(\sum_{t\in\mathcal{I}}\frac{\ln\alpha_t}{\alpha_t}-K_{\mathcal{I}}\left(\sum_{t=0}^{T}\bar{c}_t\right)\right)\right]=\hat{\lambda}\left(-\bar{c}\right)$$

by (5.11). This means

$$\hat{\lambda}\left(c-\delta\mathbb{1}-\bar{c}\right)\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t}-|\mathcal{I}|=\hat{\lambda}\left(-\bar{c}\right)\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t}-|\mathcal{I}|\,,$$

in other words,

$$V\left(c-\delta\mathbb{1}-\bar{c}\right)=V\left(-\bar{c}\right)$$

(Theorem 5.5). Notice that $V\left(-\bar{c}\right)$ is finite. Therefore, by Proposition 3.40, we have

$$\pi_{\mathrm{F}}^{ai}\left(c;\bar{c}\right)=\delta=K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}\bar{c}_t\right)-K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}(\bar{c}_t-c_t)\right).$$

Combining this with (3.53), it follows that

$$\pi_{\mathrm{F}}^{bi}\left(c;\bar{c}\right)=-\pi_{\mathrm{F}}^{ai}\left(-c;\bar{c}\right)=K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}(\bar{c}_t+c_t)\right)-K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}\bar{c}_t\right).$$

This completes the proof. □

Remark 5.8. Notice that, from Theorem 5.7, we always have

$$\pi_{\mathrm{F}}^{ai}\left(0;\bar{c}\right)=0\text{ for all }\bar{c}\in\mathcal{N}^2,$$

and this agrees with Example 3.42.

The following result shows that sometimes the indifference prices are the same for two different endowments.

Corollary 5.9. *Let $c,\bar{c},\bar{c}'\in\mathcal{N}^2$. We have*

$$\pi_{\mathrm{F}}^{ai}(c;\bar{c})=\pi_{\mathrm{F}}^{ai}(c;\bar{c}'),\qquad\qquad \pi_{\mathrm{F}}^{bi}(c;\bar{c})=\pi_{\mathrm{F}}^{bi}(c;\bar{c}')\tag{5.15}$$

if one of the following condition is satisfied.

1. *The processes $\bar{c},\bar{c}'$ satisfy $\sum_{t=0}^{T}\bar{c}_t=\sum_{t=0}^{T}\bar{c}'_t+(\delta,0)$ for some $\delta\in\mathbb{R}$.*

2. *The bid-ask prices at time 0 satisfy $S_0^b=S_0^a$, and the processes $\bar{c},\bar{c}'$ satisfy $\sum_{t=0}^{T}\bar{c}_t=\sum_{t=0}^{T}\bar{c}'_t+d$ for some $d\in\mathbb{R}^2$.*

Proof. Suppose that the the condition under the first item is satisfied. It follows from (5.8) that

$$\begin{aligned}K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}\bar{c}_t\right)&=K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}\bar{c}'_t\right)+\delta,\\ K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}(\bar{c}_t-c_t)\right)&=K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}(\bar{c}'_t-c_t)\right)+\delta,\\ K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}(\bar{c}_t+c_t)\right)&=K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^{T}(\bar{c}'_t+c_t)\right)+\delta.\end{aligned}$$

Then Theorem 5.7 implies that (5.15) holds true.

Suppose that the condition under the second item holds true. Notice that, for all $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$, the martingale property of $S = (S_t)_{t=0}^T$ gives

$$\mathbb{E}_{\mathbb{Q}}[(1, S_T) \cdot d] = (1, S_0) \cdot d = (1, S_0^b) \cdot d.$$

Combining this with (5.6) and (5.7), it follows that

$$K_{\mathcal{I}}(X + d) = K_{\mathcal{I}}(X) + (1, S_0^b) \cdot d.$$

By taking $X = \sum_{t=0}^T \vec{c}'_t$, $\sum_{t=0}^T (\vec{c}'_t - c_t)$, $\sum_{t=0}^T (\vec{c}'_t + c_t)$ respectively, we have

$$\begin{aligned} K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^T \bar{c}_t\right) &= K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^T \vec{c}'_t\right) + (1, S_0^b) \cdot d, \\ K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^T (\bar{c}_t - c_t)\right) &= K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^T (\vec{c}'_t - c_t)\right) + (1, S_0^b) \cdot d, \\ K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^T (\bar{c}_t + c_t)\right) &= K_{\mathcal{I}}\left(\textstyle\sum_{t=0}^T (\vec{c}'_t + c_t)\right) + (1, S_0^b) \cdot d. \end{aligned}$$

Thus (5.15) follows from Theorem 5.7. $\square$

In the following one-step toy model, we can solve the problem (5.7) explicitly. Moreover, we will compute the indifference prices of a European call option by using the formulae from Theorem 5.7.

Example 5.10. Let $T = 1$ and $\Omega_1 = \{u, d\}$. Consider the following one-step model with transaction cost parameter $k \in [0, 1)$.

$$\begin{array}{ccc} & & S_1^{au} = 115(1+k) \\ & \overset{p}{\nearrow} & S_1^{bu} = 115(1-k) \\ S_0^a = 100 & & \\ S_0^b = 100 & & \\ & \underset{1-p}{\searrow} & S_1^{ad} = 90(1+k) \\ & & S_1^{bd} = 90(1-k) \end{array}$$

Observe that $S_0^b = S_0^a$ which means that there is no transaction costs at time 0. We assume $S_0^b > S_1^{ad}$ and $S_0^a < S_1^{bu}$. This implies that there is no overlap among the three intervals $[S_0^b, S_0^a] = \{S_0^b\}$, $[S_1^{bd}, S_1^{ad}]$, and $[S_1^{bu}, S_1^{au}]$. Clearly, the robust no-arbitrage condition is satisfied. Moreover, the market probability is given by $\mathbb{P}(u) = p$ and $\mathbb{P}(d) = 1 - p$, where $p \in (0, 1)$. Define the friction-free stock prices $(\bar{S}_0, \bar{S}_1)$ as $\bar{S}_0 = 100$, $\bar{S}_1^u = 115$, and $\bar{S}_1^d = 90$, in other words,

$$\left(\bar{S}_0, \bar{S}_1\right) = \left(S_0^b, \frac{1}{1-k} S_1^b\right) = \left(S_0^a, \frac{1}{1+k} S_1^a\right).$$

Moreover, we define $c = (c_t)_{t=0}^1$ as $c_0 = 0$ and $c_1 = (D, 0)$, where

$$D = \max\left(\bar{S}_1 - 100, 0\right) = \begin{cases} 15 & \text{on } u, \\ 0 & \text{on } d. \end{cases}$$

Observe that $c_0 + c_1 = (D, 0)$ can be regarded as the payoff of a European call option based on the friction-free prices $(\bar{S}_0, \bar{S}_1)$ and delivered by cash with strike price 100. The investor's endowment $\bar{c} = (\bar{c}_t)_{t=0}^1$ is set to be $\bar{c}_0 = \bar{c}_1 = 0$. Moreover, let $\mathcal{I} = \{0, 1\}$ and $\alpha_0 = \alpha_1 = \alpha$ for some $\alpha > 0$.

For the convenience of later calculations, let

$$\begin{aligned} \mathcal{Q} &:= \left\{ \mathbb{Q}(u) \,\middle|\, (\mathbb{Q}, S) \in \bar{\mathcal{P}} \right\} \\ &= \left\{ \mathbb{Q}(u) \,\middle|\, S_1 \in \mathcal{L}_1,\ S_1^b \le S_1 \le S_1^a,\ \mathbb{Q}(u) S_1^u + (1 - \mathbb{Q}(u))\, S_1^d = S_0^b \right\} \\ &= \left\{ q \in [0,1] \,\middle|\, x_1 \in \left[S_1^{bu}, S_1^{au}\right], x_2 \in \left[S_1^{bd}, S_1^{ad}\right], q x_1 + (1 - q)\, x_2 = S_0^b \right\}. \end{aligned}$$

Observe from Lemma A.7 that $\mathcal{Q}$ is convex. Then $\mathcal{Q}$ is a subinterval of $[0, 1]$. Since there is no overlap between $[S_1^{bu}, S_1^{au}]$ and $[S_1^{bd}, S_1^{ad}]$, we can write $\mathcal{Q}$ as

$$\mathcal{Q} = \left\{ q \in [0,1] \,\middle|\, x_1 \in \left[S_1^{bu}, S_1^{au}\right], x_2 \in \left[S_1^{bd}, S_1^{ad}\right], q = \frac{S_0^b - x_2}{x_1 - x_2} \right\}. \tag{5.16}$$

By straightforward calculation, the value $\frac{S_0^b - x_2}{x_1 - x_2}$ in (5.16) is decreasing in x_1 (resp. x_2) when x_2 (resp. x_1) is fixed. Thus, by letting

$$\left[q^{\min}, q^{\max}\right] := \left[\frac{S_0^b - S_1^{ad}}{S_1^{au} - S_1^{ad}}, \frac{S_0^b - S_1^{bd}}{S_1^{bu} - S_1^{bd}}\right] \subseteq (0, 1),$$

it follows that $\mathcal{Q} = [q^{\min}, q^{\max}]$.

From Theorem 5.7, the indifference prices of c can be written as

$$\pi_{\mathrm{F}}^{ai}(c; 0) = K_{\mathcal{I}}(0) - K_{\mathcal{I}}(-c_0 - c_1) = K_{\mathcal{I}}(0) - K_{\mathcal{I}}(-(D, 0)), \tag{5.17}$$
$$\pi_{\mathrm{F}}^{bi}(c; 0) = K_{\mathcal{I}}(c_0 + c_1) - K_{\mathcal{I}}(0) = K_{\mathcal{I}}((D, 0)) - K_{\mathcal{I}}(0). \tag{5.18}$$

In order to compute $\pi_{\mathrm{F}}^{ai}(c; 0)$ and $\pi_{\mathrm{F}}^{bi}(c; 0)$, we are going to find the following three values:

$$\begin{aligned} K_{\mathcal{I}}(0) &= \inf_{(\mathbb{Q}, S) \in \bar{\mathcal{P}}} H_{\mathcal{I}}\left((\mathbb{Q}, S); 0\right), \\ K_{\mathcal{I}}((D, 0)) &= \inf_{(\mathbb{Q}, S) \in \bar{\mathcal{P}}} H_{\mathcal{I}}\left((\mathbb{Q}, S); (D, 0)\right), \\ K_{\mathcal{I}}((-D, 0)) &= \inf_{(\mathbb{Q}, S) \in \bar{\mathcal{P}}} H_{\mathcal{I}}\left((\mathbb{Q}, S); (-D, 0)\right). \end{aligned}$$

For any $(\mathbb{Q}, S) \in \bar{\mathcal{P}}$ and $x \in \mathcal{L}_1$, we have

$$\begin{aligned} H_{\mathcal{I}}\left((\mathbb{Q}, S);(x, 0)\right) &= \frac{1}{\alpha}\mathbb{E}\left[\Lambda_0^{\mathbb{Q}} \ln \Lambda_0^{\mathbb{Q}}\right] + \frac{1}{\alpha}\mathbb{E}\left[\Lambda_1^{\mathbb{Q}} \ln \Lambda_1^{\mathbb{Q}}\right] + \mathbb{E}_{\mathbb{Q}}\left[x\right] \\ &= \frac{1}{\alpha}\left(\mathbb{E}\left[\Lambda_1^{\mathbb{Q}} \ln \Lambda_1^{\mathbb{Q}}\right] + \alpha \mathbb{E}_{\mathbb{Q}}\left[x\right]\right) \end{aligned} \tag{5.19}$$

because $\frac{1}{\alpha}\mathbb{E}[\Lambda_0^{\mathbb{Q}} \ln \Lambda_0^{\mathbb{Q}}] = 0$ (see (2.18)). By letting

$$f^x(q) := q \ln \frac{q}{p} + (1-q) \ln \frac{1-q}{1-p} + \alpha\left(q x^u + (1-q)\, x^d\right) \text{ for all } q \in (0,1),$$

the value $\mathbb{E}[\Lambda_1^{\mathbb{Q}} \ln \Lambda_1^{\mathbb{Q}}] + \alpha \mathbb{E}_{\mathbb{Q}}\left[x\right]$ in (5.19) can be written as

$$\begin{aligned} &\mathbb{E}\left[\Lambda_1^{\mathbb{Q}} \ln \Lambda_1^{\mathbb{Q}}\right] + \alpha \mathbb{E}_{\mathbb{Q}}\left[x\right] \\ &= \mathbb{Q}(u) \ln \frac{\mathbb{Q}(u)}{\mathbb{P}(u)} + (1-\mathbb{Q}(u)) \ln \frac{1-\mathbb{Q}(u)}{1-\mathbb{P}(u)} + \alpha\left(\mathbb{Q}(u) x^u + (1-\mathbb{Q}(u))\, x^d\right) \\ &= f^x(\mathbb{Q}\,(u)). \end{aligned} \tag{5.20}$$

Then (5.19) and (5.20) imply

$$\begin{aligned} \inf_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} H_{\mathcal{I}}\left((\mathbb{Q}, S);(x, 0)\right) &= \frac{1}{\alpha} \inf_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} f^x(\mathbb{Q}(u)) \\ &= \frac{1}{\alpha} \inf_{q \in [q^{\min}, q^{\max}]} f^x(q). \end{aligned} \tag{5.21}$$

For any $q \in (0,1)$, the derivatives $f^{x\prime}(q)$ and $f^{x\prime\prime}(q)$ are

$$\begin{aligned} f^{x\prime}(q) &= \ln \frac{q}{p} - \ln \frac{1-q}{1-p} + \alpha\left(x^u - x^d\right) = \ln \frac{q}{1-q} - \ln \frac{p}{1-p} + \alpha\left(x^u - x^d\right), \\ f^{x\prime\prime}(q) &= \frac{1}{q} + \frac{1}{1-q} > 0. \end{aligned}$$

Thus f^x is continuous and convex. Let

$$q^x = \frac{p e^{-\alpha x^u}}{p e^{-\alpha x^u} + (1-p) e^{-\alpha x^d}} \in (0,1).$$

Observe that $q^0 = p$. Moreover, we have

$$\ln \frac{q^x}{1-q^x} = \ln \frac{p e^{-\alpha x^u}}{(1-p) e^{-\alpha x^d}} = \ln \frac{p}{1-p} - \alpha\left(x^u - x^d\right),$$

and hence $f^{x\prime}(q^x) = 0$. Since $f^{x\prime}$ is increasing on $(0,1)$, we have $f^{x\prime} < 0$ on $(0, q^x)$ and $f^{x\prime} > 0$ on $(q^x, 1)$. Thus the continuous function f^x is decreasing on $(0, q^x]$ and increasing on $[q^x, 1)$. We can conclude that the function f^x on

	$k = 0.5\%$		$k = 1\%$	
α	buyer	seller	buyer	seller
	regret indifference prices			
	$p = 0.3$			
0.1	5.70149	6.12470	5.40594	6.25197
0.01	5.70149	5.70149	5.40594	5.40594
	$p = 0.5$			
0.1	5.86361	6.30151	5.72975	6.60606
0.01	6.30151	6.30151	6.60606	6.60606
	superhedging prices			
	5.70149	6.30151	5.40594	6.60606

Table 5.1: Option prices of a call option in a one-step model

$[q^{\min}, q^{\max}]$ reaches its minimum at

$$\hat{q}^x := \begin{cases} q^x & \text{if } q^{\min} \leq q^x \leq q^{\max}, \\ q^{\min} & \text{if } q^x < q^{\min}, \\ q^{\max} & \text{if } q^x > q^{\max}. \end{cases}$$

Thus

$$K_{\mathcal{I}}((x,0)) = \inf_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} H_{\mathcal{I}}\left((\mathbb{Q},S);(x,0)\right) = \frac{1}{\alpha} f^x\left(\hat{q}^x\right)$$

by (5.21). By taking $x = 0, (D,0), (-D,0)$ respectively, we have

$$\begin{aligned} K_{\mathcal{I}}(0) &= \frac{1}{\alpha} f^0\left(\hat{q}^0\right), \\ K_{\mathcal{I}}((D,0)) &= \frac{1}{\alpha} f^D\left(\hat{q}^D\right), \\ K_{\mathcal{I}}(-(D,0)) &= \frac{1}{\alpha} f^{-D}\left(\hat{q}^{-D}\right). \end{aligned}$$

Thus, we are able to compute the prices in (5.17)-(5.18).

From (2.26), (2.27), and Theorem 2.14, the seller's and buyer's superhedging prices $\pi^a_{\mathrm{F}}(c)$ and $\pi^b_{\mathrm{F}}(c)$ are given by

$$\begin{aligned} \pi^a_{\mathrm{F}}(c) = \pi^a_{\mathrm{E}}((D,0)) &= \max_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}}[D], \\ \pi^b_{\mathrm{F}}(c) = \pi^b_{\mathrm{E}}((D,0)) &= \min_{(\mathbb{Q},S)\in\bar{\mathcal{P}}} \mathbb{E}_{\mathbb{Q}}[D]. \end{aligned}$$

Combining this with the fact that

$$\mathbb{E}_{\mathbb{Q}}[D] = \mathbb{Q}(u)D^u + (1-\mathbb{Q}(u))D^d \text{ for all } (\mathbb{Q},S) \in \bar{\mathcal{P}},$$

it follows that

$$\pi^a_{\mathrm{F}}(c) = \max_{q\in[q^{\min},q^{\max}]} \left(qD^u + (1-q)D^d\right) = q^{\max} \times 15,$$
$$\pi^b_{\mathrm{F}}(c) = \min_{q\in[q^{\min},q^{\max}]} \left(qD^u + (1-q)D^d\right) = q^{\min} \times 15,$$

Therefore, the superhedging prices $\pi^a_{\mathrm{F}}(c)$ and $\pi^b_{\mathrm{F}}(c)$ can be easily calculated.

In Table 5.1, the regret indifference prices for $p = 0.3, 0.5$, $k = 0.5\%, 1\%$, and $\alpha = 0.1, 0.01$ are provided. Moreover, the superhedging prices are also given in this table for $k = 0.5\%, 1\%$. It shows that the gap between seller's and buyer's indifference prices is smaller than the gap between seller's and buyer's superhedging prices. For the buyer (resp. seller), the indifference price is equal to the superhedging price when $p = 0.3$ (resp. $p = 0.5$). In the case when $\alpha = 0.01$, the seller's and buyer's indifference prices are the same.

This section ends with the proof of Lemma 5.3.

Proof of Lemma 5.3. We assume that $(\hat{\mathbb{Q}}, \hat{S}) \in \bar{\mathcal{P}}$ solves (5.7). Suppose by contradiction that $\hat{\mathbb{Q}}(\omega') = 0$ for some $\omega' \in \Omega$. From the comments following Theorem 2.6, we have $\mathcal{P} \neq \emptyset$. Let $(\mathbb{Q}, S) \in \mathcal{P}$. Then $\mathbb{Q} \sim \mathbb{P}$ which means $\mathbb{Q}(\omega) > 0$ for all $\omega \in \Omega$. For every $\epsilon \in (0,1)$, we are going to define a pair $(\mathbb{Q}^\epsilon, S^\epsilon) \in \mathcal{P}$ based on $(\mathbb{Q}, S)$ and $(\hat{\mathbb{Q}}, \hat{S})$. Let

$$\mathbb{Q}^\epsilon(A) := \epsilon\mathbb{Q}(A) + (1-\epsilon)\hat{\mathbb{Q}}(A) \text{ for all } A \in \mathcal{F}.$$

Then $\mathbb{Q}^\epsilon$ is a probability measure with $\mathbb{Q}^\epsilon(\omega) > 0$ for all $\omega \in \Omega$, and hence $\mathbb{Q}^\epsilon \sim \mathbb{P}$. Moreover, for any $t = 0, \ldots, T$ and $\nu \in \Omega_t$, the value $\Lambda^{\mathbb{Q}^\epsilon}_t(\nu)$ can be written as

$$\Lambda^{\mathbb{Q}^\epsilon}_t(\nu) = \frac{\mathbb{Q}^\epsilon(\nu)}{\mathbb{P}(\nu)} = \frac{\epsilon\mathbb{Q}(\nu) + (1-\epsilon)\hat{\mathbb{Q}}(\nu)}{\mathbb{P}(\nu)} = \epsilon\Lambda^{\mathbb{Q}}_t(\nu) + (1-\epsilon)\Lambda^{\hat{\mathbb{Q}}}_t(\nu) > 0;$$

see (2.17) for the definition of $\Lambda^{\mathbb{Q}^\epsilon}_t$. Define $S^\epsilon = (S^\epsilon_t)^T_{t=0} \in \mathcal{N}$ as

$$S^\epsilon_t := \frac{\epsilon\Lambda^{\mathbb{Q}}_t}{\Lambda^{\mathbb{Q}^\epsilon}_t} S_t + \frac{(1-\epsilon)\Lambda^{\hat{\mathbb{Q}}}_t}{\Lambda^{\mathbb{Q}^\epsilon}_t} \hat{S}_t \text{ for all } t = 0, \ldots, T.$$

Since $\frac{\epsilon\Lambda^{\mathbb{Q}}_t}{\Lambda^{\mathbb{Q}^\epsilon}_t}$ and $\frac{(1-\epsilon)\Lambda^{\hat{\mathbb{Q}}}_t}{\Lambda^{\mathbb{Q}^\epsilon}_t}$ take their values in $[0,1]$ and $\frac{\epsilon\Lambda^{\mathbb{Q}}_t}{\Lambda^{\mathbb{Q}^\epsilon}_t} + \frac{(1-\epsilon)\Lambda^{\hat{\mathbb{Q}}}_t}{\Lambda^{\mathbb{Q}^\epsilon}_t} = 1$, we must have $S^b_t \leq S^\epsilon_t \leq S^a_t$. For any $k = 1, \ldots, T$, Bayes' formula (Shreve 2004, Lemma 5.2.2) gives

$$\Lambda^{\mathbb{Q}^\epsilon}_{k-1}\mathbb{E}_{\mathbb{Q}^\epsilon}\left[S^\epsilon_k \middle| \mathcal{F}_{k-1}\right] = \mathbb{E}\left[\Lambda^{\mathbb{Q}^\epsilon}_k S^\epsilon_k \middle| \mathcal{F}_{k-1}\right]$$

$$
\begin{aligned}
&= \epsilon \mathbb{E}\left[\Lambda_k^{\mathbb{Q}} S_k \middle| \mathcal{F}_{k-1} \right] + (1-\epsilon) \mathbb{E}\left[\Lambda_k^{\hat{\mathbb{Q}}} \hat{S}_k \middle| \mathcal{F}_{k-1} \right] \\
&= \epsilon \Lambda_{k-1}^{\mathbb{Q}} \mathbb{E}_{\mathbb{Q}}\left[S_k \right| \mathcal{F}_{k-1}] + (1-\epsilon) \Lambda_{k-1}^{\hat{\mathbb{Q}}} \mathbb{E}_{\hat{\mathbb{Q}}}\left[\hat{S}_k \middle| \mathcal{F}_{k-1} \right].
\end{aligned}
$$

Since $(S_t)_{t=0}^T$ is a $\mathbb{Q}$-martingale and $(\hat{S}_t)_{t=0}^T$ is a $\hat{\mathbb{Q}}$-martingale, we have

$$
\mathbb{E}_{\mathbb{Q}^\epsilon}\left[S_k^\epsilon \right| \mathcal{F}_{k-1}] = \frac{1}{\Lambda_{k-1}^{\mathbb{Q}^\epsilon}} \left(\epsilon \Lambda_{k-1}^{\mathbb{Q}} S_{k-1} + (1-\epsilon) \Lambda_{k-1}^{\hat{\mathbb{Q}}} \hat{S}_{k-1} \right) = S_{k-1}^\epsilon.
$$

This implies that S^ϵ is a $\mathbb{Q}^\epsilon$-martingale, and hence $(\mathbb{Q}^\epsilon, S^\epsilon) \in \mathcal{P}$.

Our next objective is to show that

$$
H_{\mathcal{I}}\left(\left(\mathbb{Q}^{\epsilon'}, S^{\epsilon'} \right); X \right) - H_{\mathcal{I}}\left(\left(\hat{\mathbb{Q}}, \hat{S} \right); X \right) < 0 \text{ for some } \epsilon' \in (0,1), \tag{5.22}
$$

which contradicts the assumption that $(\hat{\mathbb{Q}}, \hat{S}) \in \bar{\mathcal{P}}$ solves (5.7). For convenience, let $g(x) := x \ln x$ for all $x \geq 0$, where $0 \ln 0 \equiv 0$. Notice from $\hat{\mathbb{Q}}(\omega') = 0$ that

$$
\begin{aligned}
\lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left[g\left(\Lambda_T^{\mathbb{Q}^\epsilon}(\omega') \right) - g\left(\Lambda_T^{\hat{\mathbb{Q}}}(\omega') \right) \right] &= \lim_{\epsilon \downarrow 0} \frac{1}{\epsilon} \left[g\left(\frac{\epsilon \mathbb{Q}(\omega')}{\mathbb{P}(\omega')} \right) - g(0) \right] \\
&= \lim_{\epsilon \downarrow 0} \frac{\mathbb{Q}(\omega')}{\mathbb{P}(\omega')} \ln \frac{\epsilon \mathbb{Q}(\omega')}{\mathbb{P}(\omega')} = -\infty.
\end{aligned} \tag{5.23}
$$

Fix any $\epsilon \in (0,1)$. For each $t = 0, \ldots, T$, the convexity of g gives

$$
g\left(\Lambda_t^{\mathbb{Q}^\epsilon} \right) \leq \epsilon g\left(\Lambda_t^{\mathbb{Q}} \right) + (1-\epsilon) g\left(\Lambda_t^{\hat{\mathbb{Q}}} \right).
$$

By subtracting $g(\Lambda_t^{\hat{\mathbb{Q}}})$ on both sides, we have

$$
g\left(\Lambda_t^{\mathbb{Q}^\epsilon} \right) - g\left(\Lambda_t^{\hat{\mathbb{Q}}} \right) \leq \epsilon \left[g\left(\Lambda_t^{\mathbb{Q}} \right) - g\left(\Lambda_t^{\hat{\mathbb{Q}}} \right) \right]. \tag{5.24}
$$

Notice that

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}^\epsilon}\left[(1, S_T^\epsilon) \cdot X \right] &= \mathbb{E}\left[\Lambda_T^{\mathbb{Q}^\epsilon} (1, S_T^\epsilon) \cdot X \right], \\
\mathbb{E}_{\hat{\mathbb{Q}}}\left[\left(1, \hat{S}_T \right) \cdot X \right] &= \mathbb{E}\left[\Lambda_T^{\hat{\mathbb{Q}}} \left(1, \hat{S}_T \right) \cdot X \right].
\end{aligned}
$$

Combining this with (5.6), it follows that

$$
\begin{aligned}
&H_{\mathcal{I}}\left((\mathbb{Q}^\epsilon, S^\epsilon); X \right) - H_{\mathcal{I}}\left(\left(\hat{\mathbb{Q}}, \hat{S} \right); X \right) \\
=& \sum_{t \in \mathcal{I}} \frac{1}{\alpha_t} \mathbb{E}\left[g\left(\Lambda_t^{\mathbb{Q}^\epsilon} \right) - g\left(\Lambda_t^{\hat{\mathbb{Q}}} \right) \right] + \mathbb{E}\left[\left(\Lambda_T^{\mathbb{Q}^\epsilon} (1, S_T^\epsilon) - \Lambda_T^{\hat{\mathbb{Q}}} (1, \hat{S}_T) \right) \cdot X \right] \\
=& \epsilon \left(\sum_{t \in \mathcal{I}} \frac{1}{\epsilon \alpha_t} \mathbb{E}\left[g\left(\Lambda_t^{\mathbb{Q}^\epsilon} \right) - g\left(\Lambda_t^{\hat{\mathbb{Q}}} \right) \right] + \frac{1}{\epsilon} \mathbb{E}\left[\left(\Lambda_T^{\mathbb{Q}^\epsilon} (1, S_T^\epsilon) - \Lambda_T^{\hat{\mathbb{Q}}} (1, \hat{S}_T) \right) \cdot X \right] \right)
\end{aligned}
$$

$$=\epsilon\left(m^{\epsilon}+\frac{1}{\epsilon\alpha_T}\mathbb{E}\left[\mathbf{1}_{\{\omega'\}}\left(g\left(\Lambda_T^{\mathbb{Q}^{\epsilon}}\right)-g\left(\Lambda_T^{\hat{\mathbb{Q}}}\right)\right)\right]\right), \tag{5.25}$$

where

$$m^{\epsilon}:=\frac{1}{\epsilon}\mathbb{E}\left[\left(\Lambda_T^{\mathbb{Q}^{\epsilon}}(1,S_T^{\epsilon})-\Lambda_T^{\hat{\mathbb{Q}}}(1,\hat{S}_T)\right)\cdot X\right]$$
$$+\sum_{t\in\mathcal{I}\setminus\{T\}}\frac{1}{\epsilon\alpha_t}\mathbb{E}\left[g\left(\Lambda_t^{\mathbb{Q}^{\epsilon}}\right)-g\left(\Lambda_t^{\hat{\mathbb{Q}}}\right)\right]+\frac{1}{\epsilon\alpha_T}\mathbb{E}\left[\mathbf{1}_{\Omega\setminus\{\omega'\}}\left(g\left(\Lambda_T^{\mathbb{Q}^{\epsilon}}\right)-g\left(\Lambda_T^{\hat{\mathbb{Q}}}\right)\right)\right].$$

We are going to show that m^{ϵ} is dominated by some $M\in\mathbb{R}$ that is independent of ϵ. Define $M^1,M^2,M^3\in\mathbb{R}$ as

$$M^1:=\mathbb{E}\left[\left(\Lambda_T^{\mathbb{Q}}-\Lambda_T^{\hat{\mathbb{Q}}},\Lambda_T^{\mathbb{Q}}S_T-\Lambda_T^{\hat{\mathbb{Q}}}\hat{S}_T\right)\cdot X\right],$$
$$M^2:=\sum_{t\in\mathcal{I}\setminus\{T\}}\frac{1}{\alpha_t}\mathbb{E}\left[g\left(\Lambda_t^{\mathbb{Q}}\right)-g\left(\Lambda_t^{\hat{\mathbb{Q}}}\right)\right],$$
$$M^3:=\frac{1}{\alpha_T}\mathbb{E}\left[\mathbf{1}_{\Omega\setminus\{\omega'\}}\left(g\left(\Lambda_T^{\mathbb{Q}}\right)-g\left(\Lambda_T^{\hat{\mathbb{Q}}}\right)\right)\right].$$

The quantities M^1,M^2,M^3 are independent of ϵ. Now, observe that

$$\begin{aligned}\Lambda_T^{\mathbb{Q}^{\epsilon}}(1,S_T^{\epsilon})&=\left(\epsilon\Lambda_T^{\mathbb{Q}}+(1-\epsilon)\Lambda_T^{\hat{\mathbb{Q}}},\epsilon\Lambda_T^{\mathbb{Q}}S_T+(1-\epsilon)\Lambda_T^{\hat{\mathbb{Q}}}\hat{S}_T\right)\\&=\epsilon\left(\Lambda_T^{\mathbb{Q}}-\Lambda_T^{\hat{\mathbb{Q}}},\Lambda_T^{\mathbb{Q}}S_T-\Lambda_T^{\hat{\mathbb{Q}}}\hat{S}_T\right)+\Lambda_T^{\hat{\mathbb{Q}}}\left(1,\hat{S}_T\right).\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{\epsilon}\mathbb{E}\left[\left(\Lambda_T^{\mathbb{Q}^{\epsilon}}(1,S_T^{\epsilon})-\Lambda_T^{\hat{\mathbb{Q}}}(1,\hat{S}_T)\right)\cdot X\right]&=\frac{1}{\epsilon}\mathbb{E}\left[\epsilon\left(\Lambda_T^{\mathbb{Q}}-\Lambda_T^{\hat{\mathbb{Q}}},\Lambda_T^{\mathbb{Q}}S_T-\Lambda_T^{\hat{\mathbb{Q}}}\hat{S}_T\right)\cdot X\right]\\&=M^1.\end{aligned}$$

Moreover, it follows from (5.24) that

$$\sum_{t\in\mathcal{I}\setminus\{T\}}\frac{1}{\epsilon\alpha_t}\mathbb{E}\left[g\left(\Lambda_t^{\mathbb{Q}^{\epsilon}}\right)-g\left(\Lambda_t^{\hat{\mathbb{Q}}}\right)\right]\leq\sum_{t\in\mathcal{I}\setminus\{T\}}\frac{\epsilon}{\epsilon\alpha_t}\mathbb{E}\left[g\left(\Lambda_t^{\mathbb{Q}}\right)-g\left(\Lambda_t^{\hat{\mathbb{Q}}}\right)\right]=M^2$$

and

$$\begin{aligned}\frac{1}{\epsilon\alpha_T}\mathbb{E}\left[\mathbf{1}_{\Omega\setminus\{\omega'\}}\left(g\left(\Lambda_T^{\mathbb{Q}^{\epsilon}}\right)-g\left(\Lambda_T^{\hat{\mathbb{Q}}}\right)\right)\right]&\leq\frac{\epsilon}{\epsilon\alpha_T}\mathbb{E}\left[\mathbf{1}_{\Omega\setminus\{\omega'\}}\left(g\left(\Lambda_T^{\mathbb{Q}}\right)-g\left(\Lambda_T^{\hat{\mathbb{Q}}}\right)\right)\right]\\&=M^3.\end{aligned}$$

Thus, by letting $M=M^1+M^2+M^3$, it follows that $m^{\epsilon}\leq M$. From (5.23), there exists $\epsilon' \in (0,1)$ such that

$$\frac{1}{\epsilon'\alpha_T}\mathbb{E}\left[\mathbf{1}_{\{\omega'\}}\left(g\left(\Lambda_T^{\mathbb{Q}^{\epsilon'}}\right) - g\left(\Lambda_T^{\hat{\mathbb{Q}}}\right)\right)\right] < -M.$$

Then (5.25) implies

$$\begin{aligned} H_{\mathcal{I}}\left(\left(\mathbb{Q}^{\epsilon'}, S^{\epsilon'}\right); X\right) - H_{\mathcal{I}}\left(\left(\hat{\mathbb{Q}}, \hat{S}\right); X\right) \\ &= \epsilon'\left(m^{\epsilon'} + \frac{1}{\epsilon'\alpha_T}\mathbb{E}\left[\mathbf{1}_{\{\omega'\}}\left(g\left(\Lambda_T^{\mathbb{Q}^{\epsilon'}}\right) - g\left(\Lambda_T^{\hat{\mathbb{Q}}}\right)\right)\right]\right) \\ &\leq \epsilon'\left(M + \frac{1}{\epsilon'\alpha_T}\mathbb{E}\left[\mathbf{1}_{\{\omega'\}}\left(g\left(\Lambda_T^{\mathbb{Q}^{\epsilon'}}\right) - g\left(\Lambda_T^{\hat{\mathbb{Q}}}\right)\right)\right]\right) < 0. \end{aligned}$$

This completes the proof of (5.22). Therefore $\hat{\mathbb{Q}}(\omega) > 0$ for all $\omega \in \Omega$, in other words, we have $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$. □

5.2 Existence of a solution to the dual problem

In this section, we are going to study the minimisation problem (5.7). The main objective of this section is to construct a solution to (5.7). Firstly, we will assume that the bid-ask prices S^b and S^a satisfy (5.26) for the remainder of this chapter. Then Section 5.2.1 will introduce the notion of transition probabilities and provides a number of technical results that will be used in Section 5.2.2. After that, a dynamic programming algorithm will be provided in Section 5.2.2 to construct a solution to the minimisation problem (5.7).

For each $t = 0, \ldots, T-1$ and $\nu \in \Omega_t$, the *collection of successor nodes of* ν is defined as

$$\nu^+ := \{\lambda \in \Omega_{t+1} | \lambda \subseteq \nu\}.$$

Fix any $t = 0, \ldots, T$ and $\nu \in \Omega_t$. For any $k = 0, \ldots, t$, we define ν_k as the unique node in Ω_k such that $\nu \subseteq \nu_k$. Moreover, in the case when $k < t$, we call ν_k the *predecessor node of* ν at time step k. Notice that the node at time t that contains $\omega \in \Omega$ can be written as $\{\omega\}_t$.

For the remainder of this chapter, we shall always assume that the bid-ask prices S^b and S^a satisfy

$$\min_{\lambda \in \nu^+} S_{t+1}^{b\lambda} < S_t^{b\nu} \leq S_t^{a\nu} < \max_{\lambda \in \nu^+} S_{t+1}^{a\lambda} \text{ for all } t = 0, \ldots, T-1, \nu \in \Omega_t. \quad (5.26)$$

This assumption means that the bid price at time t and node ν is higher than the minimal bid price at time $t+1$ among every node $\lambda \in \nu^+$. Similarly, the ask price at time t and node ν is lower than the maximal ask price at time $t+1$ among every node $\lambda \in \nu^+$. All numerical examples in this chapter will satisfy (5.26).

It turns out that the condition (5.26) implies that the robust no-arbitrage condition introduced in Theorem 2.6 holds true; see Theorem 5.13. In the following example, we will provide an one-step model such that the robust no-arbitrage condition is satisfied but (5.26) is not satisfied.

Example 5.11. Consider a market model with $T = 1$ and $\Omega_1 = \{u, d\}$. The stock prices $(S_t^b, S_t^a)_{t=0}^1$ are given by

$$S_0^b = S_1^b = 90,\ S_0^a = S_1^a = 120.$$

Clearly, the condition (5.26) above is not satisfied. Now, we define a process $S = (S_t)_{t=0}^1 \in \mathcal{N}$ as

$$S_0 = 100,\ S_1^u = 105,\ S_1^d = 95.$$

Observe that S_t is in the relative interior of $[S_t^b, S_t^a]$ for each $t = 0, 1$. Then we define a probability $\mathbb{Q}$ as

$$\mathbb{Q}(u) = \mathbb{Q}(d) = \frac{1}{2}.$$

Clearly, the process S is a $\mathbb{Q}$-martingale, which means $(\mathbb{Q}, S) \in \mathcal{P}$. Thus, the robust no-arbitrage condition is satisfied.

5.2.1 Transition probability

In this section, we will first introduce the notion of transition probabilities. Then Theorem 5.13 shows that (5.26) implies the robust no-arbitrage condition. After that, we will provide a number of technical results in Lemmas 5.14-5.16 for the study in the next section.

Let $\mathbb{Q}$ be a probability measure. For each $t = 0, \dots, T$, let

$$\Omega_t^+(\mathbb{Q}) := \{\nu \in \Omega_t | \, \mathbb{Q}(\nu) > 0\}$$

be the collection of nodes at time t with positive probability under $\mathbb{Q}$. Fix any $t = 0, \dots T-1$. For every $\nu \in \Omega_t^+(\mathbb{Q})$ and $\lambda \in \nu^+$, we denote the *transition probability* of ν to λ by

$$q_{t+1}^\lambda := \frac{\mathbb{Q}(\lambda)}{\mathbb{Q}(\nu)}.$$

For any $Y \in \mathcal{L}_{t+1}$, we can present $\mathbb{E}_\mathbb{Q}[Y \mid \mathcal{F}_t](\nu)$ for each $\nu \in \Omega_t^+(\mathbb{Q})$ as

$$\mathbb{E}_\mathbb{Q}[Y \mid \mathcal{F}_t](\nu) = \frac{\sum_{\lambda \in \nu^+} \mathbb{Q}(\lambda) Y^\lambda}{\mathbb{Q}(\nu)} = \sum_{\lambda \in \nu^+} q_{t+1}^\lambda Y^\lambda. \tag{5.27}$$

Remark 5.12. The expectation $\mathbb{E}_{\mathbb{Q}}[Y]$ can be presented as

$$\mathbb{E}_{\mathbb{Q}}[Y] = \sum_{\nu\in\Omega_t^+(\mathbb{Q})} \mathbb{Q}(\nu) \sum_{\lambda\in\nu^+} q_{t+1}^{\lambda} Y^{\lambda}. \tag{5.28}$$

Indeed, we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[Y] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}[Y \mid \mathcal{F}_t]\right] \\
&= \sum_{\nu\in\Omega_t} \mathbb{Q}(\nu)\mathbb{E}_{\mathbb{Q}}[Y \mid \mathcal{F}_t](\nu) \\
&= \sum_{\nu\in\Omega_t^+(\mathbb{Q})} \mathbb{Q}(\nu)\mathbb{E}_{\mathbb{Q}}[Y \mid \mathcal{F}_t](\nu)
\end{aligned}$$

because $\mathbb{Q}(\nu) = 0$ for any $\nu \in \Omega_t\backslash\Omega_t^+(\mathbb{Q})$. Combining this with (5.27), the presentation of $\mathbb{E}_{\mathbb{Q}}[Y]$ in (5.28) holds true. The formulation of expectation in (5.28) will be used in later calculations.

The following result says that (5.26) implies the robust no-arbitrage condition introduced in Theorem 2.6.

Theorem 5.13. *Under the assumption that* (5.26) *holds true, the robust no-arbitrage condition holds true.*

Proof. Firstly, we are going to construct a process $S = (S_t)_{t=0}^T \in \mathcal{N}$. Let

$$S_0 := \frac{1}{2}\left(S_0^b + S_0^a\right) \in \operatorname{relint}\left[S_0^b, S_0^a\right],$$

where relint A is the relative interior of a set A. For any $t = 0, \dots, T-1$ and $\nu \in \Omega_t$, we define λ' and λ'' as the nodes in ν^+ such that

$$\begin{aligned}
S_{t+1}^{b\lambda'} &= \min_{\lambda\in\nu^+} S_{t+1}^{b\lambda}, \\
S_{t+1}^{a\lambda''} &= \max_{\lambda\in\nu^+} S_{t+1}^{a\lambda}.
\end{aligned}$$

Moreover, we define S_{t+1}^{λ} for each $\lambda \in \nu^+$ as

$$\begin{aligned}
S_{t+1}^{\lambda} &= \frac{1}{2}\left(S_{t+1}^{b\lambda} + S_{t+1}^{a\lambda}\right) \text{ for all } \lambda \in \nu^+\backslash\left\{\lambda', \lambda''\right\}, \\
S_{t+1}^{\lambda'} &= \frac{1}{2}\left(S_{t+1}^{b\lambda'} + \min\left(S_t^{b\nu}, S_{t+1}^{a\lambda'}\right)\right), \\
S_{t+1}^{\lambda''} &= \frac{1}{2}\left(S_{t+1}^{a\lambda''} + \max\left(S_t^{a\nu}, S_{t+1}^{b\lambda''}\right)\right),
\end{aligned}$$

where $S_{t+1}^{b\lambda'} < S_t^{b\nu}$ and $S_{t+1}^{a\lambda''} > S_t^{a\nu}$ by (5.26). Then

$$S_{t+1}^{\lambda} \in \operatorname{relint}\left[S_{t+1}^{b\lambda}, S_{t+1}^{a\lambda}\right] \text{ for all } \lambda \in \nu^+$$

and moreover

$$\min_{\lambda\in\nu^+} S^\lambda_{t+1} \le S^{\lambda'}_{t+1} < S^{b\nu}_t \le S^{a\nu}_t < S^{\lambda''}_{t+1} \le \max_{\lambda\in\nu^+} S^\lambda_{t+1}. \tag{5.29}$$

This completes the construction of S.

For any $t = 0,\dots,T-1$ and $\nu \in \Omega_t$, the construction of S^ν_t gives

$$S^{b\nu}_t \le S^\nu_t \le S^{a\nu}_t.$$

Combining this with (5.29), it follows that

$$\min_{\lambda\in\nu^+} S^\lambda_{t+1} < S^\nu_t < \max_{\lambda\in\nu^+} S^\lambda_{t+1}.$$

Thus, there exists a collection of positive quantities $(w^\lambda_{t+1})_{\lambda\in\nu^+}$ in $(0,1)$ such that

$$\sum_{\lambda\in\nu^+} w^\lambda_{t+1} = 1,$$
$$\sum_{\lambda\in\nu^+} w^\lambda_{t+1} S^\lambda_{t+1} = S^\nu_t.$$

Let $\mathbb{Q} \sim \mathbb{P}$ be the probability measure such that

$$q^\lambda_{t+1} = w^\lambda_{t+1} \text{ for all } t = 0,\dots,T-1, \nu \in \Omega_t, \lambda \in \nu^+.$$

Observe that, for each $t = 0,\dots,T-1$ and $\nu \in \Omega_t$, we have $\nu \in \Omega^+_t(\mathbb{Q})$ and

$$\mathbb{E}_\mathbb{Q}\left[S_{t+1} \mid \mathcal{F}_t\right](\nu) = \sum_{\lambda\in\nu^+} q^\lambda_{t+1} S^\lambda_{t+1} = \sum_{\lambda\in\nu^+} w^\lambda_{t+1} S^\lambda_{t+1} = S^\nu_t.$$

Thus S is a $\mathbb{Q}$-martingale. Combining this with the fact that S_t is in the relative interior of $[S^b_t, S^a_t]$ for every $t = 0,\dots,T$, we have $(\mathbb{Q}, S) \in \mathcal{P}$. Moreover, the robust no-arbitrage condition introduced in Theorem 2.6 holds true. This completes the proof. □

The results in Lemmas 5.14-5.16 below will be helpful in the next section. For any probability $\mathbb{Q}$ and $\mathbb{Q}$-martingale $(M_t)^T_{t=0}$, the result below gives a link between $\mathbb{E}_\mathbb{Q}[M_{t+1} \ln \Lambda^\mathbb{Q}_{t+1}]$ and $\mathbb{E}_\mathbb{Q}[M_t \ln \Lambda^\mathbb{Q}_t]$ for any $t = 0,\dots,T-1$.

Lemma 5.14. *Let $\mathbb{Q}$ be a probability measure and $(M_t)^T_{t=0}$ be a $\mathbb{Q}$-martingale. Then, for each $t = 0,\dots,T-1$, we have*

$$\mathbb{E}_\mathbb{Q}\left[M_{t+1} \ln \Lambda^\mathbb{Q}_{t+1}\right] = \mathbb{E}_\mathbb{Q}\left[M_t \ln \Lambda^\mathbb{Q}_t\right] + \sum_{\nu\in\Omega^+_t(\mathbb{Q})} \mathbb{Q}(\nu) \sum_{\lambda\in\nu^+} M^\lambda_{t+1} q^\lambda_{t+1} \ln \frac{q^\lambda_{t+1}}{p^\lambda_{t+1}}.$$

Proof. Let $t = 0, \dots, T-1$. Observe from Remark 5.12 that

$$\mathbb{E}_{\mathbb{Q}}\left[M_{t+1}\ln\Lambda^{\mathbb{Q}}_{t+1}\right] = \sum_{\nu\in\Omega^+_t(\mathbb{Q})}\mathbb{Q}(\nu)\sum_{\lambda\in\nu^+}q^\lambda_{t+1}M^\lambda_{t+1}\ln\frac{\mathbb{Q}(\nu)q^\lambda_{t+1}}{\mathbb{P}(\nu)p^\lambda_{t+1}}$$
$$= \sum_{\nu\in\Omega^+_t(\mathbb{Q})}\mathbb{Q}(\nu)\ln\frac{\mathbb{Q}(\nu)}{\mathbb{P}(\nu)}\sum_{\lambda\in\nu^+}q^\lambda_{t+1}M^\lambda_{t+1}+\sum_{\nu\in\Omega^+_t(\mathbb{Q})}\mathbb{Q}(\nu)\sum_{\lambda\in\nu^+}M^\lambda_{t+1}q^\lambda_{t+1}\ln\frac{q^\lambda_{t+1}}{p^\lambda_{t+1}}.$$

The martingale property of $(M_t)_{t=0}^T$ gives

$$\begin{aligned}\sum_{\nu\in\Omega^+_t(\mathbb{Q})}\mathbb{Q}(\nu)\ln\frac{\mathbb{Q}(\nu)}{\mathbb{P}(\nu)}\sum_{\lambda\in\nu^+}q^\lambda_{t+1}M^\lambda_{t+1} &= \sum_{\nu\in\Omega^+_t(\mathbb{Q})}\mathbb{Q}(\nu)M^\nu_t\ln\frac{\mathbb{Q}(\nu)}{\mathbb{P}(\nu)}\\ &= \mathbb{E}_{\mathbb{Q}}\left[M_t\ln\Lambda^{\mathbb{Q}}_t\right].\end{aligned}$$

Then the result follows. □

For any $t = 0, \dots, T$, let

$$\begin{aligned}\bar{\mathcal{P}}_t := \Big\{\Big(\mathbb{Q},(S_k)_{k=0}^t\Big)\Big|\,&\mathbb{Q}\text{ a probability measure, } S_k\in\mathcal{L}_k\ \forall k=0,\dots,t,\\ &\exists\Big(\mathbb{Q}^*,(S^*_k)_{k=0}^T\Big)\in\bar{\mathcal{P}}:\ \mathbb{Q}^*=\mathbb{Q}\text{ on }\mathcal{F}_t,\ (S^*_k)_{k=0}^t=(S_k)_{k=0}^t\Big\}.\end{aligned}\tag{5.30}$$

Observe that $\bar{\mathcal{P}}_T = \bar{\mathcal{P}}$.

Lemma 5.15. *Let $t = 0, \dots, T$. Moreover, let $\mathbb{Q}$ be a probability measure and $S_k \in \mathcal{L}_k$ for all $k = 0, \dots, t$. Then $(\mathbb{Q}, (S_k)_{k=0}^t) \in \bar{\mathcal{P}}_t$ if and only if*

$$S^b_i \le S_i \le S^a_i \text{ for all } i = 0,\dots,t, \tag{5.31}$$
$$\textstyle\sum_{\lambda\in\nu^+}q^\lambda_{i+1}S^\lambda_{i+1} = S^\nu_i \text{ for all } i=0,\dots,t-1, \nu\in\Omega^+_i(\mathbb{Q}). \tag{5.32}$$

Proof. Suppose that $(\mathbb{Q}, (S_k)_{k=0}^t) \in \bar{\mathcal{P}}_t$. Then there exists $(\mathbb{Q}^*, (S^*_k)_{k=0}^T) \in \bar{\mathcal{P}}$ such that $\mathbb{Q}^* = \mathbb{Q}$ on $\mathcal{F}_t$ and $(S^*_k)_{k=0}^t = (S_k)_{k=0}^t$. Since $(S^*_k)_{k=0}^t = (S_k)_{k=0}^t$ and

$$S^b_i \le S^*_i \le S^a_i \text{ for all } i = 0,\dots,t,$$

the condition (5.31) holds true. For any $i = 0, \dots, t-1$ and $\nu \in \Omega^+_i(\mathbb{Q})$, we have $\nu \in \Omega^+_i(\mathbb{Q}^*)$, and the martingale property of $(S^*_k)_{k=0}^T$ gives

$$\sum_{\lambda\in\nu^+}q^{*\lambda}_{i+1}S^{*\lambda}_{i+1} = S^{*\nu}_i.$$

Thus

$$\sum_{\lambda\in\nu^+}q^\lambda_{i+1}S^\lambda_{i+1} = \sum_{\lambda\in\nu^+}q^{*\lambda}_{i+1}S^{*\lambda}_{i+1} = S^{*\nu}_i = S^\nu_i.$$

Therefore (5.32) holds true.

Suppose that (5.31) and (5.32) hold true. We are going to prove

$$(\mathbb{Q}, (S_k)_{k=0}^{t}) \in \bar{\mathcal{P}}_t \tag{5.33}$$

by considering the following the following two cases of t.

In the case when $t = T$, we have

$$\mathbb{E}\left[S_{k+1} \middle| \mathcal{F}_k\right](\nu) = \sum_{\lambda \in \nu^+} q_{k+1}^{\lambda} S_{k+1}^{\lambda} = S_k^{\nu} \text{ for all } k = 0, \dots, T-1, \nu \in \Omega_k^+(\mathbb{Q}).$$

This means that $(S_k)_{k=0}^{T}$ is a $\mathbb{Q}$-martingale. Combining this and the fact that (5.31) holds true for $t = T$, it follows that

$$(\mathbb{Q}, (S_k)_{k=0}^{T}) \in \bar{\mathcal{P}} = \bar{\mathcal{P}}_T,$$

which proves (5.33).

In the second case, we assume that $t < T$. For each $k = t, \dots, T-1$, we are going to define $S_{k+1} \in \mathcal{L}_{k+1}$ as follows. For every $\nu \in \Omega_k$, let $\lambda', \lambda'' \in \nu^+$ such that

$$\begin{aligned} S_{k+1}^{b\lambda'} &= \min_{\lambda \in \nu^+} S_{k+1}^{b\lambda}, \\ S_{k+1}^{a\lambda''} &= \max_{\lambda \in \nu^+} S_{k+1}^{a\lambda}. \end{aligned}$$

Then we define $(S_{k+1}^{\lambda})_{\lambda \in \nu^+}$ as

$$S_{k+1}^{\lambda} := \begin{cases} S_{k+1}^{b\lambda'} & \text{if } \lambda = \lambda', \\ S_{k+1}^{a\lambda''} & \text{if } \lambda = \lambda'', \\ \frac{1}{2}\left(S_{k+1}^{b\lambda} + S_{k+1}^{a\lambda}\right) & \text{if } \lambda \in \nu \backslash \{\lambda', \lambda''\}. \end{cases}$$

Notice that

$$S_{k+1}^{\lambda} \in \left[S_{k+1}^{b\lambda}, S_{k+1}^{a\lambda}\right] \text{ for all } \lambda \in \nu^+,$$

and

$$\min_{\lambda \in \nu^+} S_{k+1}^{\lambda} = S_{k+1}^{\lambda'} < S_k^{b\nu} \leq S_k^{a\nu} < S_{k+1}^{\lambda''} = \max_{\lambda \in \nu^+} S_{k+1}^{\lambda}. \tag{5.34}$$

This completes the definition of S_{k+1}. Notice that $(S_k)_{k=0}^{T} \in \mathcal{N}$ and

$$S_k^b \leq S_k \leq S_k^a \text{ for all } k = 0, \dots, T.$$

For any $k = t, \dots, T-1$, and $\nu \in \Omega_k$, combining $S_k^{b\nu} \leq S_k^{\nu} \leq S_k^{a\nu}$ with (5.34), it follows that

$$\min_{\lambda\in\nu^+} S^\lambda_{k+1} < S^\nu_k < \max_{\lambda\in\nu^+} S^\lambda_{k+1}.$$

Then there exists a collection of positive numbers $(w^\lambda_{k+1})_{\lambda\in\nu^+}$ in $(0,1)$ such that

$$\sum_{\lambda\in\nu^+} w^\lambda_{k+1} = 1,$$
$$\sum_{\lambda\in\nu^+} w^\lambda_{k+1} S^\lambda_{k+1} = S^\nu_k.$$

Let $\mathbb{Q}^* \ll \mathbb{P}$ be the probability measure such that $\mathbb{Q}^* = \mathbb{Q}$ on $\mathcal{F}_t$ and

$$q^{*\lambda}_{k+1} = w^\lambda_{k+1} \text{ for all } k = t,\dots,T-1, \nu\in\Omega^+_k(\mathbb{Q}^*), \lambda\in\nu^+. \tag{5.35}$$

Such $\mathbb{Q}^*$ can be constructed by specifying its transition probabilities with the values of $(w_{k+1})^{T-1}_{k=t}$ via (5.35). Moreover, since the values of $(w_{k+1})^{T-1}_{k=t}$ are always positive, the family $\Omega^+_k(\mathbb{Q}^*)$ in (5.35) can be written as

$$\begin{aligned}\Omega^+_k(\mathbb{Q}^*) &= \left\{\nu\in\Omega_k \,\middle|\, \nu_t\in\Omega^+_t(\mathbb{Q}^*)\right\}\\ &= \left\{\nu\in\Omega_k \,\middle|\, \nu_t\in\Omega^+_t(\mathbb{Q})\right\}.\end{aligned}$$

Let

$$(S^*_k)^T_{k=0} := (S_k)^T_{k=0}.$$

Then

$$S^b_k \le S^*_k \le S^a_k \text{ for all } k=0,\dots,T.$$

In addition, by straightforward calculation, it follows that

$$\mathbb{E}\left[S^*_{k+1}\middle|\mathcal{F}_k\right](\nu) = \sum_{\lambda\in\nu^+} q^{*\lambda}_{k+1} S^{*\lambda}_{k+1} = S^{*\nu}_k \text{ for all } k=0,\dots,T-1, \nu\in\Omega^+_k(\mathbb{Q}).$$

Thus $(S^*_k)^T_{k=0}$ is a $\mathbb{Q}^*$-martingale, and hence

$$(\mathbb{Q}^*, (S^*_k)^T_{k=0}) \in \bar{\mathcal{P}}.$$

Therefore (5.33) holds true. This completes the proof. □

For all $0\le t\le t'\le T$ and $(\mathbb{Q}, S)\in\bar{\mathcal{P}}_t$, we define

$$\bar{\mathcal{P}}^t_{t'}(\mathbb{Q},S) := \left\{(\mathbb{Q}', (S'_k)^{t'}_{k=0}) \in \bar{\mathcal{P}}_{t'} \,\middle|\, \mathbb{Q}'=\mathbb{Q} \text{ on } \mathcal{F}_t, (S'_k)^t_{k=0} = S\right\} \subseteq \bar{\mathcal{P}}_{t'}. \tag{5.36}$$

Fix any $t=0,\dots,T-1$ and $(\mathbb{Q},S) = (\mathbb{Q},(S_k)^t_{k=0}) \in \bar{\mathcal{P}}_t$. Let F_{t+1} be an

$\mathbb{R} \cup \{\infty\}$-valued $\mathcal{F}_{t+1}$-measurable random function on $\Omega \times \mathbb{R}^2$ such that the function $\omega \mapsto F^{\omega}_{t+1}$ is constant on each node in Ω_{t+1}. Moreover, for every $\lambda \in \Omega_{t+1}$, we assume that F^{λ}_{t+1} is bounded from below and $F^{\lambda}_{t+1}(q,s) \in \mathbb{R}$ for all $q \in [0,1]$ and $s \in [S^{b\lambda}_{t+1}, S^{a\lambda}_{t+1}]$. Let

$$\mathcal{V}_t(\mathbb{Q}, S) := \inf_{(\mathbb{Q}', (S'_k)_{k=0}^{t+1}) \in \bar{\mathcal{P}}^t_{t+1}(\mathbb{Q}, S)} \left\{ \sum_{\nu \in \Omega^+_t(\mathbb{Q}')} \mathbb{Q}'(\nu) \sum_{\lambda \in \nu^+} F^{\lambda}_{t+1}\left(q'^{\lambda}_{t+1}, S'^{\lambda}_{t+1}\right) \right\}. \tag{5.37}$$

In the problem (5.37), for each $\nu \in \Omega^+_t(\mathbb{Q}')$, we have $\mathbb{Q}'(\nu) = \mathbb{Q}(\nu)$ because $\mathbb{Q}' = \mathbb{Q}$ on $\mathcal{F}_t$. Moreover, the control variables $(q'^{\lambda}_{t+1})_{\lambda \in \nu^+}$ are transition probabilities which means that $q'^{\lambda}_{t+1} \in [0,1]$ for all $\lambda \in \nu^+$ and $\sum_{\lambda \in \nu^+} q'^{\lambda}_{t+1} = 1$. In addition, the control variables $(S'^{\lambda}_{t+1})_{\lambda \in \nu^+}$ satisfies $S'^{\lambda}_{t+1} \in [S^{b\lambda}_{t+1}, S^{a\lambda}_{t+1}]$ for all $\lambda \in \nu^+$ and $\sum_{\lambda \in \nu^+} q'^{\lambda}_{t+1} S'^{\lambda}_{t+1} = S'^{\nu}_t = S^{\nu}_t$ by Lemma 5.15 and $S'_t = S_t$. The result below says that the problem (5.37) above can be decoupled into multiple minimisation problems.

Lemma 5.16. *For every $t = 0, \ldots, T-1$ and $(\mathbb{Q}, S) = (\mathbb{Q}, (S_k)_{k=0}^t) \in \bar{\mathcal{P}}_t$, we have*

$$\begin{aligned} \mathcal{V}_t(\mathbb{Q}, S) = \sum_{\nu \in \Omega^+_t(\mathbb{Q})} \mathbb{Q}(\nu) \inf \Bigg\{ \sum_{\lambda \in \nu^+} F^{\lambda}_{t+1}\left(w^{\lambda}_{t+1}, s^{\lambda}_{t+1}\right) \Bigg| \, & w^{\lambda}_{t+1} \in [0,1], \\ s^{\lambda}_{t+1} \in \left[S^{b\lambda}_{t+1}, S^{a\lambda}_{t+1}\right] \ \forall \lambda \in \nu^+, \sum_{\lambda \in \nu^+} w^{\lambda}_{t+1} = 1, \sum_{\lambda \in \nu^+} w^{\lambda}_{t+1} s^{\lambda}_{t+1} = S^{\nu}_t & \Bigg\}. \end{aligned}$$

Proof. It follows from (5.37) and the comments following it that

$$\begin{aligned} \mathcal{V}_t(\mathbb{Q}, S) \geq \sum_{\nu \in \Omega^+_t(\mathbb{Q})} \mathbb{Q}(\nu) \inf \Bigg\{ \sum_{\lambda \in \nu^+} F^{\lambda}_{t+1}\left(w^{\lambda}_{t+1}, s^{\lambda}_{t+1}\right) \Bigg| \, & w^{\lambda}_{t+1} \in [0,1], \\ s^{\lambda}_{t+1} \in \left[S^{b\lambda}_{t+1}, S^{a\lambda}_{t+1}\right] \ \forall \lambda \in \nu^+, \sum_{\lambda \in \nu^+} w^{\lambda}_{t+1} = 1, \sum_{\lambda \in \nu^+} w^{\lambda}_{t+1} s^{\lambda}_{t+1} = S^{\nu}_t & \Bigg\}. \end{aligned} \tag{5.38}$$

We are going to show that the opposite inequality of (5.38) also holds true. Suppose that $(w^{\lambda}_{t+1}, s^{\lambda}_{t+1})_{\lambda \in \nu^+}$ is a collection of quantities that satisfies the constraints in (5.38) for every $\nu \in \Omega^+_t(\mathbb{Q})$. Then, for any $\nu \in \Omega^+_t(\mathbb{Q})$, we have $w^{\lambda}_{t+1} \in [0,1]$ and $s^{\lambda}_{t+1} \in [S^{b\lambda}_{t+1}, S^{a\lambda}_{t+1}]$ for all $\lambda \in \nu^+$, and moreover

$$\sum_{\lambda \in \nu^+} w^{\lambda}_{t+1} = 1,$$
$$\sum_{\lambda \in \nu^+} w^{\lambda}_{t+1} s^{\lambda}_{t+1} = S^{\nu}_t.$$

Let $\mathbb{Q}' \ll \mathbb{P}$ be a probability measure such that $\mathbb{Q}' = \mathbb{Q}$ on $\mathcal{F}_t$ and that

$$q'^{\lambda}_{t+1} = w^{\lambda}_{t+1} \text{ for all } \lambda \in \nu^+ \text{ and } \nu \in \Omega^+_t(\mathbb{Q}).$$

In addition, we define $(S'_k)_{k=0}^{t+1}$ as

$$\begin{aligned}(S'_k)_{k=0}^{t} =& (S_k)_{k=0}^{t} = S,\\ S'_{t+1} =& s_{t+1}.\end{aligned}$$

Then $(\mathbb{Q}', (S'_k)_{k=0}^{t+1}) \in \bar{\mathcal{P}}_{t+1}$ by Lemma 5.15. Combining this with $\mathbb{Q}' = \mathbb{Q}$ on $\mathcal{F}_t$ and $(S'_k)_{k=0}^{t} = S$, it follows that

$$\left(\mathbb{Q}', (S'_k)_{k=0}^{t+1}\right) \in \bar{\mathcal{P}}^t_{t+1}(\mathbb{Q}, S).$$

Moreover, we have

$$\sum_{\nu \in \Omega^+_t(\mathbb{Q}')} \mathbb{Q}'(\nu) \sum_{\lambda \in \nu^+} F^{\lambda}_{t+1}\left(q'^{\lambda}_{t+1}, S'^{\lambda}_{t+1}\right) = \sum_{\nu \in \Omega^+_t(\mathbb{Q})} \mathbb{Q}(\nu) \sum_{\lambda \in \nu^+} F^{\lambda}_{t+1}\left(w^{\lambda}_{t+1}, s^{\lambda}_{t+1}\right).$$

Thus, the definition of $\mathcal{V}_t(\mathbb{Q}, S)$ in (5.37) gives

$$\mathcal{V}_t(\mathbb{Q}, S) \leq \sum_{\nu \in \Omega^+_t(\mathbb{Q})} \mathbb{Q}(\nu) \sum_{\lambda \in \nu^+} F^{\lambda}_{t+1}\left(w^{\lambda}_{t+1}, s^{\lambda}_{t+1}\right).$$

By taking infimum on both sides, the opposite inequality of (5.38) holds true, and hence the result follows. □

5.2.2 Construction of a solution to the dual problem

In this section, Algorithm 5.17 first constructs a sequence of random functions $(J_t)_{t=0}^T$. Then Proposition 5.18 provides a number of properties of $(J_t)_{t=0}^T$, and it shows that every optimisation problem in Algorithm 5.17 admits a solution. Based on $(J_t)_{t=0}^T$, we will construct a pair $(\hat{\mathbb{Q}}, \hat{S})$ in Algorithm 5.19. Then Theorem 5.20 shows that $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$ solves the problem (5.7), and that the optimal value of (5.7) is closely related to J_0. In order to prove Theorem 5.20, we will provides a number of technical results in Propositions 5.21-5.22. The proof of Theorem 5.20 will be provided at the end of the section.

For convenience, for every $t = 1, \ldots, T$, let

$$l_t := \sum_{k \in \mathcal{I}, k \geq t} \frac{1}{\alpha_k} \tag{5.39}$$

be the accumulated value of the quantities $(\frac{1}{\alpha_k})_{k \in \mathcal{I}, k \geq t}$; the quantity α_0 is not used in defining l_t. Observe that $l_T = \frac{1}{\alpha_T}$ because $T \in \mathcal{I}$. Moreover, for any sequence $n_1, \dots, n_T \in \mathbb{R}$, we have

$$\begin{aligned}
\sum_{t\in\mathcal{I}\setminus\{0\}} \frac{1}{\alpha_t}\sum_{k=1}^{t} n_k &= \sum_{t\in\mathcal{I}\setminus\{0\}} \left(\frac{1}{\alpha_t}n_1 + \cdots + \frac{1}{\alpha_t}n_t\right) \\
&= \sum_{k\in\mathcal{I},k\geq 1}\frac{1}{\alpha_k}n_1 + \cdots + \sum_{k\in\mathcal{I},k\geq T}\frac{1}{\alpha_k}n_T \\
&= l_1 n_1 + \cdots + l_T n_T \\
&= \sum_{k=1}^{T} l_k n_k. \qquad (5.40)
\end{aligned}$$

The following algorithm constructs a sequence of random functions $(J_t)_{t=0}^T$. For each $t = 0, \dots, T$, the random function J_t will be $\mathcal{F}_t$-measurable. It turns out that the minimal value of J_0 is equal to $K_{\mathcal{I}}(X)$; see Algorithm 5.19 and Theorem 5.20. Moreover, the random functions $(J_t)_{t=0}^T$ will be used in Algorithm 5.19 to construct a pair $(\hat{\mathbb{Q}}, \hat{S})$.

Algorithm 5.17. *Construct a sequence of random functions $(J_t)_{t=0}^T$.*

We are going to define random function J_t recursively for each $t = T, \dots, 0$. For every $\nu \in \Omega_T$ and $s \in \mathbb{R}$, let

$$J_T^\nu(s) := \begin{cases} (1,s)\cdot X^\nu = X^{b\nu} + sX^{s\nu} & \text{if } s \in \left[S_T^{b\nu}, S_T^{a\nu}\right], \\ \infty & \text{otherwise.} \end{cases} \qquad (5.41)$$

Let $t = T-1, \dots, 0$ and $\nu \in \Omega_t$. Define $J_t^\nu : \mathbb{R} \mapsto \mathbb{R}\cup\{\infty\}$ as follows. For all $s \in [S_t^{b\nu}, S_t^{a\nu}]$, let

$$\begin{aligned}
J_t^\nu(s) := l_{t+1} \inf_{(q_{t+1}^\lambda, s_{t+1}^\lambda)_{\lambda\in\nu^+}} &\left\{ \sum_{\lambda\in\nu^+} q_{t+1}^\lambda \left(\ln\frac{q_{t+1}^\lambda}{p_{t+1}^\lambda} + \frac{J_{t+1}^\lambda(s_{t+1}^\lambda)}{l_{t+1}} \right) \middle| \, q_{t+1}^\lambda \in [0,1], \right. \\
&\left. s_{t+1}^\lambda \in \left[S_{t+1}^{b\lambda}, S_{t+1}^{a\lambda}\right] \; \forall \lambda \in \nu^+, \sum_{\lambda\in\nu^+} q_{t+1}^\lambda = 1, \sum_{\lambda\in\nu^+} q_{t+1}^\lambda s_{t+1}^\lambda = s \right\}. \qquad (5.42)
\end{aligned}$$

Moreover, let

$$J_t^\nu = \infty \text{ on } \mathbb{R}\setminus\left[S_t^{b\nu}, S_t^{a\nu}\right].$$

This completes the construction of $(J_t)_{t=0}^T$.

It turns out that there always exists a solution to the problem (5.42); see Proposition 5.18.2. This means that the infimum in (5.42) is always attained. In Section 5.3, we will discuss the approximation of $(J_t)_{t=0}^T$.

In addition to the existence of solution of the problem (5.42), the proposition below also provides a number of properties of $(J_t)_{t=0}^T$. Some results established in Sections 4.1 and 4.2 will be used in the proof of the following result.

Proposition 5.18. *The following claims hold true.*

1. *Let $t = 0, \dots, T$. For any $\nu \in \Omega_t$, the function J_t^ν is real-valued, convex and continuous on* $\operatorname{dom} J_t^\nu = [S_t^{b\nu}, S_t^{a\nu}]$.

2. *There always exists a solution to the minimisation problem in* (5.42).

3. *Let $t = 0, \dots, T$. For any $\nu \in \Omega_t$, the function $s \mapsto J_t^\nu(s)$ is Lipschitz continuous on* $\operatorname{dom} J_t^\nu$*, in other words, there exists $A_t^\nu \in [0, \infty)$ such that*

$$|J_t^\nu(s_1) - J_t^\nu(s_2)| \leq A_t^\nu |s_1 - s_2| \textit{ for all } s_1, s_2 \in \left[S_t^{b\nu}, S_t^{a\nu}\right].$$

Proof. Firstly, we are going to prove by backward induction that the first claim holds true. For any $\nu \in \Omega_T$, the function J_T^ν is affine on $\operatorname{dom} J_T^\nu = [S_T^{b\nu}, S_T^{a\nu}]$. Thus, the conditions on J_t in the first claim holds true for $t = T$. Fix any $i = 0, \dots, T-1$. Suppose that the conditions on J_t in the first claim holds true for $t = i+1$. This implies that, for any $\nu \in \Omega_i$, the function J_{i+1}^λ is real-valued, convex and continuous on $\operatorname{dom} J_{i+1}^\lambda = [S_{i+1}^{b\lambda}, S_{i+1}^{a\lambda}]$ for all $\lambda \in \nu^+$. Moreover, the condition (5.26) gives

$$\left[S_i^{b\nu}, S_i^{a\nu}\right] \subseteq \operatorname{co}\left(\bigcup_{\lambda \in \nu^+} \left[S_{i+1}^{b\lambda}, S_{i+1}^{a\lambda}\right]\right) = \operatorname{co}\left(\bigcup_{\lambda \in \nu^+} \operatorname{dom} J_{i+1}^\lambda\right),$$

where $\operatorname{co}(A)$ is the convex hull of a set A. From Theorems 4.3 and 4.13, the function J_i^ν is real-valued, convex and continuous on $[S_i^{b\nu}, S_i^{a\nu}]$. The construction of J_i^ν in Algorithm 5.17 gives

$$J_i^\nu = \infty \text{ on } \mathbb{R} \backslash [S_i^{b\nu}, S_i^{a\nu}],$$

and hence $\operatorname{dom} J_i^\nu = [S_i^{b\nu}, S_i^{a\nu}]$. Therefore, the conditions on J_t in the first claim holds true for $t = i$. This completes the induction step, and hence the first claim holds true.

The second claim follows from Theorem 4.13.

It remains to prove the third claim. Fix any $\nu \in \Omega_T$. The function J_T^ν is affine on $[S_T^{b\nu}, S_T^{a\nu}]$ with slope $X^{s\nu}$. Then, by letting $A_T^\nu := |X^{s\nu}| \geq 0$, it follows that

$$|J_T^\nu(s_1) - J_T^\nu(s_2)| = A_T^\nu |s_1 - s_2| \text{ for all } s_1, s_2 \in \left[S_T^{b\nu}, S_T^{a\nu}\right].$$

Fix any $t = 0, \dots, T-1$ and $\nu \in \Omega_t$. We are going to define a function $s \mapsto J_t^{*\nu}(s)$ that satisfies

$$J_t^{*\nu} = J_t^{\nu} \text{ on } \left[S_t^{b\nu}, S_t^{a\nu}\right] = \operatorname{dom} J_t^{\nu}.$$

For any $s \in \mathbb{R}$, let

$$J_t^{*\nu}(s) := l_{t+1} \inf_{(q_{t+1}^{\lambda}, s_{t+1}^{\lambda})_{\lambda \in \nu^+}} \left\{ \sum_{\lambda \in \nu^+} q_{t+1}^{\lambda} \left(\ln \frac{q_{t+1}^{\lambda}}{p_{t+1}^{\lambda}} + \frac{J_{t+1}^{\lambda}(s_{t+1}^{\lambda})}{l_{t+1}} \right) \right| \\ \left. q_{t+1}^{\lambda} \in [0,1], s_{t+1}^{\lambda} \in [S_{t+1}^{b\lambda}, S_{t+1}^{a\lambda}] \; \forall \lambda \in \nu^+, \sum_{\lambda \in \nu^+} q_{t+1}^{\lambda} = 1, \sum_{\lambda \in \nu^+} q_{t+1}^{\lambda} s_{t+1}^{\lambda} = s \right\}.$$

Theorem 4.3 implies that $J_t^{*\nu}$ is $\mathbb{R} \cup \{\infty\}$-valued, convex, and

$$\operatorname{dom} J_t^{*\nu} = \operatorname{co}\left(\bigcup_{\lambda \in \nu^+} \left[S_{t+1}^{b\lambda}, S_{t+1}^{a\lambda}\right] \right) = \left[\min_{\lambda \in \nu^+} S_{t+1}^{b\lambda}, \max_{\lambda \in \nu^+} S_{t+1}^{a\lambda} \right].$$

Moreover, it follows from Theorem 4.13 that $J_t^{*\nu}$ is continuous on $\operatorname{dom} J_t^{*\nu}$. From (5.26), we have

$$\min_{\lambda \in \nu^+} S_{t+1}^{b\lambda} < S_t^{b\nu} \le S_t^{a\nu} < \max_{\lambda \in \nu^+} S_{t+1}^{a\lambda}.$$

This means

$$\operatorname{int}\left(\operatorname{dom} J_t^{*\nu}\right) \supseteq \left[S_t^{b\nu}, S_t^{a\nu}\right] = \operatorname{dom} J_t^{\nu},$$

where $\operatorname{int} B$ is the interior of any set B. Then there exists $A_t^{\nu} \in [0, \infty)$ such that

$$\left|J_t^{\nu}(s_1) - J_t^{\nu}(s_2)\right| = \left|J_t^{*\nu}(s_1) - J_t^{*\nu}(s_2)\right| \le A_t^{\nu} \left|s_1 - s_2\right| \text{ for all } s_1, s_2 \in \operatorname{dom} J_t^{\nu}$$

(Rockafellar 1997, Theorem 24.7). This completes the proof. □

The algorithm below will introduce a pair $(\hat{\mathbb{Q}}, \hat{S})$ that relies on the sequence of random functions $(J_t)_{t=0}^T$ constructed in Algorithm 5.17 above. It turns out that $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$ and this pair is a solution to the minimisation problem (5.7); see Theorem 5.20.

Algorithm 5.19. *Construct a pair $(\hat{\mathbb{Q}}, \hat{S}) = (\hat{\mathbb{Q}}, (\hat{S}_t)_{t=0}^T)$.*

Firstly, we shall construct $\bar{q}_t, \bar{s}_t \in \mathcal{L}_t$ recursively for each $t = 0, \dots, T$. Let $\bar{q}_0 = 1$, and let $\bar{s}_0 \in [S_0^b, S_0^a]$ such that

$$J_0(\bar{s}_0) = \inf_{s \in [S_0^b, S_0^a]} J_0(s).$$

For each $t = 0, \ldots, T-1$ and $\nu \in \Omega_t$, let $(\bar{q}^{\lambda}_{t+1}, \bar{s}^{\lambda}_{t+1})_{\lambda \in \nu^+}$ be a solution to the problem in (5.42) *with $s = \bar{s}^{\nu}_t$. This completes the construction of $(\bar{q}_t, \bar{s}_t)_{t=0}^T$. Secondly, we define $(\hat{\mathbb{Q}}, \hat{S})$ as*

$$\hat{\mathbb{Q}}(A) := \sum_{\omega \in A} \prod_{t=0}^{T} \bar{q}_t^{\{\omega\}_t} \text{ for all } A \in \mathcal{F},$$
$$\hat{S}_t := \bar{s}_t \text{ for all } t = 0, \ldots, T;$$

the value of an empty summation is assumed to be 0 *(which means $\hat{\mathbb{Q}}(\emptyset) = 0$). This completes the construction.*

From Proposition 5.18.1, the function $s \mapsto J_0(s)$ is continuous on $[S_0^b, S_0^a]$. This means that $\bar{s}_0$ introduced in Algorithm 5.19 exists. Moreover, from Proposition 5.18.2, there always exists a solution to the problem in (5.42). This implies that the sequence of random variables $(\bar{q}_t, \bar{s}_t)_{t=1}^T$ constructed in Algorithm 5.19 also exist.

Theorem 5.20 below is the main result of this section. It shows that the pair $(\hat{\mathbb{Q}}, \hat{S})$ constructed in Algorithm 5.19 above is a solution to the problem (5.7). Moreover, it shows that $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$ which means $\hat{\mathbb{Q}} \sim \mathbb{P}$ (i.e. $\hat{\mathbb{Q}}(\omega) > 0$ for every $\omega \in \Omega$). In addition, this theorem also shows that $K_{\mathcal{I}}(X)$ (which is the optimal value of the minimisation problem in (5.7)) is closely related to the function J_0 constructed in Algorithm 5.17.

Theorem 5.20. *Under the assumption that* (5.26) *holds true, the pair $(\hat{\mathbb{Q}}, \hat{S}) = (\hat{\mathbb{Q}}, (\hat{S}_t)_{t=0}^T)$ constructed in Algorithm 5.19 satisfies $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$ and*

$$H_{\mathcal{I}}\left((\hat{\mathbb{Q}}, \hat{S}); X\right) = J_0(\hat{S}_0) = K_{\mathcal{I}}(X). \tag{5.43}$$

Thus $(\hat{\mathbb{Q}}, \hat{S})$ solves (5.7).

The main task of the remainder of this section is to prepare the results that will be used to prove Theorem 5.20. The proof will be provided at the end of this section.

For all $t = 0, \ldots, T$ and $(\mathbb{Q}, S) \in \bar{\mathcal{P}}_t$, let

$$V_t(\mathbb{Q}, S) := \inf_{(\mathbb{Q}^*, S^*) \in \bar{\mathcal{P}}_T^t(\mathbb{Q}, S)} H_{\mathcal{I}}\left((\mathbb{Q}^*, S^*); X\right); \tag{5.44}$$

see (5.30) for the definition of $\bar{\mathcal{P}}_t$ and see (5.36) for the definition of $\bar{\mathcal{P}}_{t'}^t(\mathbb{Q}, S)$ for all $t \leq t' \leq T$. In particular, when $t = T$, we have

$$V_T(\mathbb{Q}, S) = \inf_{(\mathbb{Q}^*, S^*) \in \bar{\mathcal{P}}_T^T(\mathbb{Q}, S)} H_{\mathcal{I}}\left((\mathbb{Q}^*, S^*); X\right) = H_{\mathcal{I}}\left((\mathbb{Q}, S); X\right) \tag{5.45}$$

because $\bar{\mathcal{P}}^T_T(\mathbb{Q},S) = \{(\mathbb{Q},S)\}$. Observe that

$$\begin{aligned}
\inf_{(\mathbb{Q},S)\in\bar{\mathcal{P}}_0} V_0(\mathbb{Q},S) &= \inf\left\{ H_{\mathcal{I}}\left((\mathbb{Q}^*,S^*);X\right) \,\middle|\, (\mathbb{Q},S)\in\bar{\mathcal{P}}_0, (\mathbb{Q}^*,S^*)\in\bar{\mathcal{P}}^0_T(\mathbb{Q},S)\right\} \\
&= \inf\left\{ H_{\mathcal{I}}\left((\mathbb{Q},S);X\right) \,\middle|\, (\mathbb{Q},S)\in\bar{\mathcal{P}}\right\} \\
&= K_{\mathcal{I}}\left(X\right) \qquad (5.46)
\end{aligned}$$

by the definition of $K_{\mathcal{I}}\left(X\right)$ in (5.7).

The following proposition provides a connection between V_t and V_{t+1} for each $t = 0,\dots,T-1$. This result is technical, and it will be used in the proof of Proposition 5.22 below.

Proposition 5.21. *Let $t = 0,\dots,T-1$ and $(\mathbb{Q},S)\in\bar{\mathcal{P}}_t$. Then*

$$V_t(\mathbb{Q},S) = \inf_{(\mathbb{Q}',S')\in\bar{\mathcal{P}}^t_{t+1}(\mathbb{Q},S)} V_{t+1}(\mathbb{Q}',S').$$

Proof. Let $t = 0,\dots,T-1$ and $(\mathbb{Q},S)\in\bar{\mathcal{P}}_t$. Notice from (5.36) that the family $\bar{\mathcal{P}}^t_T(\mathbb{Q},S)$ can be presented as

$$\begin{aligned}
\bar{\mathcal{P}}^t_T(\mathbb{Q},S) &= \left\{(\mathbb{Q}^*,(S^*_k)^T_{k=0})\in\bar{\mathcal{P}}_T \,\middle|\, \mathbb{Q}^*=\mathbb{Q} \text{ on } \mathcal{F}_t,\ (S^*_k)^t_{k=0}=S\right\} \\
&= \left\{(\mathbb{Q}^*,S^*) \,\middle|\, (\mathbb{Q}',S')\in\bar{\mathcal{P}}^t_{t+1}(\mathbb{Q},S),\ (\mathbb{Q}^*,S^*)\in\bar{\mathcal{P}}^{t+1}_T(\mathbb{Q}',S')\right\}.
\end{aligned}$$

Combining this with (5.44), we have

$$\begin{aligned}
V_t(\mathbb{Q},S) &= \inf_{(\mathbb{Q}',S')\in\bar{\mathcal{P}}^t_{t+1}(\mathbb{Q},S),\,(\mathbb{Q}^*,S^*)\in\bar{\mathcal{P}}^{t+1}_T(\mathbb{Q}',S')} H_{\mathcal{I}}\left((\mathbb{Q}^*,S^*);X\right) \\
&= \inf_{(\mathbb{Q}',S')\in\bar{\mathcal{P}}^t_{t+1}(\mathbb{Q},S)}\ \inf_{(\mathbb{Q}^*,S^*)\in\bar{\mathcal{P}}^{t+1}_T(\mathbb{Q}',S')} H_{\mathcal{I}}\left((\mathbb{Q}^*,S^*);X\right) \\
&= \inf_{(\mathbb{Q}',S')\in\bar{\mathcal{P}}^t_{t+1}(\mathbb{Q},S)} V_{t+1}(\mathbb{Q}',S').
\end{aligned}$$

This completes the proof. □

The following result gives a link between V_t and J_t where $t = 0,\dots,T$. This result will be used in the proof of Theorem 5.20.

Proposition 5.22. *For all $(\mathbb{Q},S_0)\in\bar{\mathcal{P}}_0$, we have*

$$V_0(\mathbb{Q},S_0) = J_0(S_0). \qquad (5.47)$$

Moreover, for all $t = 1,\dots,T$ and $(\mathbb{Q},(S_k)^t_{k=0})\in\bar{\mathcal{P}}_t$, we have

$$V_t(\mathbb{Q},(S_k)^t_{k=0}) = \mathbb{E}_{\mathbb{Q}}\left[J_t(S_t)\right] + \sum_{k=1}^{t} l_k\left(\mathbb{E}_{\mathbb{Q}}\left[\ln\Lambda^{\mathbb{Q}}_k\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln\Lambda^{\mathbb{Q}}_{k-1}\right]\right). \qquad (5.48)$$

Proof. Firstly, we are going to prove by backward induction that (5.48) holds true for any $(\mathbb{Q}, (S_k)_{k=0}^t) \in \bar{\mathcal{P}}_t$ for each $t = T, \dots, 1$.

For any $(\mathbb{Q}, S) = (\mathbb{Q}, (S_t)_{t=0}^T) \in \bar{\mathcal{P}} = \bar{\mathcal{P}}_T$, it follows from the construction of J_T in Algorithm 5.17 that

$$\mathbb{E}_{\mathbb{Q}}\left[(1, S_T) \cdot X\right] = \mathbb{E}_{\mathbb{Q}}\left[J_T(S_T)\right]$$

Then (5.6) gives

$$\begin{aligned} H_{\mathcal{I}}\left((\mathbb{Q}, S); X\right) &= \mathbb{E}_{\mathbb{Q}}\left[J_T(S_T)\right] + \sum_{t \in \mathcal{I}} \frac{1}{\alpha_t} \mathbb{E}\left[\Lambda_t^{\mathbb{Q}} \ln \Lambda_t^{\mathbb{Q}}\right] \\ &= \mathbb{E}_{\mathbb{Q}}\left[J_T(S_T)\right] + \sum_{t \in \mathcal{I}} \frac{1}{\alpha_t} \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_t^{\mathbb{Q}}\right]. \end{aligned}$$

Notice that

$$\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_0^{\mathbb{Q}}\right] = \mathbb{E}\left[\Lambda_0^{\mathbb{Q}} \ln \Lambda_0^{\mathbb{Q}}\right] = 0$$

by (2.18). This means that, for each $t = 1, \dots, T$, the expectation $\mathbb{E}_{\mathbb{Q}}[\ln \Lambda_t^{\mathbb{Q}}]$ can be written as

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_t^{\mathbb{Q}}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_t^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_0^{\mathbb{Q}}\right] \\ &= \sum_{k=1}^{t} \left(\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_k^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_{k-1}^{\mathbb{Q}}\right]\right). \end{aligned}$$

Combining this with $\frac{1}{\alpha_0}\mathbb{E}_{\mathbb{Q}}[\ln \Lambda_0^{\mathbb{Q}}] = 0$, we have

$$\begin{aligned} \sum_{t \in \mathcal{I}} \frac{1}{\alpha_t} \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_t^{\mathbb{Q}}\right] &= \sum_{t \in \mathcal{I}\setminus\{0\}} \frac{1}{\alpha_t} \sum_{k=1}^{t} \left(\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_k^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_{k-1}^{\mathbb{Q}}\right]\right) \\ &= \sum_{k=1}^{T} l_k \left(\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_k^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_{k-1}^{\mathbb{Q}}\right]\right) \end{aligned}$$

by (5.40). Therefore, it follows from (5.45) that

$$\begin{aligned} V_T\left(\mathbb{Q}, S\right) &= H_{\mathcal{I}}\left((\mathbb{Q}, S); X\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left[J_T(S_T)\right] + \sum_{k=1}^{T} l_k \left(\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_k^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_{k-1}^{\mathbb{Q}}\right]\right). \end{aligned}$$

Thus (5.48) holds true for $t = T$.

Let $i = 1, \dots, T-1$. Suppose that, for any $(\mathbb{Q}, (S_k)_{k=0}^{i+1}) \in \bar{\mathcal{P}}_{i+1}$, the equality (5.48) holds true for $t = i+1$. Fix any $(\mathbb{Q}, S) = (\mathbb{Q}, (S_k)_{k=0}^{i}) \in \bar{\mathcal{P}}_i$. Then we are going to show that (5.48) holds true for $t = i$. Notice that, for any $(\mathbb{Q}', (S'_k)_{k=0}^{i+1}) \in \bar{\mathcal{P}}_{i+1}^i(\mathbb{Q}, S)$, we have

$$\begin{aligned}V_{i+1}\left(\mathbb{Q}', (S'_k)_{k=0}^{i+1}\right) =& \mathbb{E}_{\mathbb{Q}'}\left[J_{i+1}(S'_{i+1})\right] + \sum_{k=1}^{i+1} l_k \left(\mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_k^{\mathbb{Q}'}\right] - \mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_{k-1}^{\mathbb{Q}'}\right]\right) \\ =& \mathbb{E}_{\mathbb{Q}'}\left[J_{i+1}(S'_{i+1})\right] + l_{i+1}\left(\mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_{i+1}^{\mathbb{Q}'}\right] - \mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_i^{\mathbb{Q}'}\right]\right) \\ &+ \sum_{k=1}^{i} l_k \left(\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_k^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_{k-1}^{\mathbb{Q}}\right]\right),\end{aligned}$$

where the second equality follows from the fact that $\mathbb{Q}' = \mathbb{Q}$ on $\mathcal{F}_i$. Moreover, from Remark 5.12, we have

$$\mathbb{E}_{\mathbb{Q}'}\left[J_{i+1}(S'_{i+1})\right] = \sum_{\nu \in \Omega_i^+(\mathbb{Q}')} \mathbb{Q}'(\nu) \sum_{\lambda \in \nu^+} q_{i+1}'^{\lambda} J_{i+1}^{\lambda}(S_{i+1}'^{\lambda}).$$

In addition, Lemma 5.14 gives

$$\mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_{i+1}^{\mathbb{Q}'}\right] - \mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_i^{\mathbb{Q}'}\right] = \sum_{\nu \in \Omega_i^+(\mathbb{Q}')} \mathbb{Q}'(\nu) \sum_{\lambda \in \nu^+} q_{i+1}'^{\lambda} \ln \frac{q_{i+1}'^{\lambda}}{p_{i+1}^{\lambda}}.$$

Thus, the value $V_{i+1}(\mathbb{Q}', (S'_k)_{k=0}^{i+1})$ can be written as

$$\begin{aligned}V_{i+1}\left(\mathbb{Q}', (S'_k)_{k=0}^{i+1}\right) = l_{i+1} \sum_{\nu \in \Omega_i^+(\mathbb{Q}')} \mathbb{Q}'(\nu) \sum_{\lambda \in \nu^+} q_{i+1}'^{\lambda} \left(\ln \frac{q_{i+1}'^{\lambda}}{p_{i+1}^{\lambda}} + \frac{J_{i+1}^{\lambda}(S_{i+1}'^{\lambda})}{l_{i+1}}\right) \\ + \sum_{k=1}^{i} l_k \left(\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_k^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_{k-1}^{\mathbb{Q}}\right]\right).\end{aligned}$$

By the connection between V_i and V_{i+1} established in Proposition 5.21, we can present $V_i(\mathbb{Q}, S)$ as

$$V_i(\mathbb{Q}, S) = \inf_{(\mathbb{Q}', (S'_k)_{k=0}^{i+1}) \in \bar{\mathcal{P}}_{i+1}^i(\mathbb{Q}, S)} V_{i+1}\left(\mathbb{Q}', (S'_k)_{k=0}^{i+1}\right).$$

Combining this with Lemma 5.16, we have

$$\begin{aligned}&V_i(\mathbb{Q}, S) \\ =& l_{i+1} \sum_{\nu \in \Omega_i^+(\mathbb{Q})} \mathbb{Q}(\nu) \inf \left\{ \sum_{\lambda \in \nu^+} w_{i+1}^{\lambda} \left(\ln \frac{w_{i+1}^{\lambda}}{p_{i+1}^{\lambda}} + \frac{J_{i+1}^{\lambda}(s_{i+1}^{\lambda})}{l_{i+1}}\right) \middle| \, w_{i+1}^{\lambda} \in [0,1], \right. \\ & \left. s_{i+1}^{\lambda} \in \left[S_{i+1}^{b\lambda}, S_{i+1}^{a\lambda}\right] \ \forall \lambda \in \nu^+, \sum_{\lambda \in \nu^+} w_{i+1}^{\lambda} = 1, \sum_{\lambda \in \nu^+} w_{i+1}^{\lambda} s_{i+1}^{\lambda} = S_i^{\nu} \right\}\end{aligned}$$

$$+\sum_{k=1}^{i} l_k \left(\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_k^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_{k-1}^{\mathbb{Q}}\right]\right).$$

Then it follows from the construction of J_i in Algorithm 5.17 that

$$\begin{aligned} V_i(\mathbb{Q}, S) = & \sum_{\nu\in\Omega_i^+(\mathbb{Q})} \mathbb{Q}(\nu) J_i^\nu(S_i^\nu) + \sum_{k=1}^{i} l_k \left(\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_k^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_{k-1}^{\mathbb{Q}}\right]\right) \\ = & \mathbb{E}_{\mathbb{Q}}\left[J_i(S_i)\right] + \sum_{k=1}^{i} l_k \left(\mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_k^{\mathbb{Q}}\right] - \mathbb{E}_{\mathbb{Q}}\left[\ln \Lambda_{k-1}^{\mathbb{Q}}\right]\right). \end{aligned}$$

This implies that (5.48) holds true for $t = i$. This completes the induction step. Therefore, we can conclude that (5.48) holds true for every $t = 1, \ldots, T$ and $(\mathbb{Q}, (S_k)_{k=0}^t) \in \bar{\mathcal{P}}_t$.

Fix any $(\mathbb{Q}, S_0) \in \bar{\mathcal{P}}_0$. We are going to show that (5.47) holds true. For any $(\mathbb{Q}', (S_k')_{k=0}^1) \in \bar{\mathcal{P}}_1^0(\mathbb{Q}, S_0)$, it follows from (5.48) that

$$V_1\left(\mathbb{Q}', (S_k')_{k=0}^1\right) = \mathbb{E}_{\mathbb{Q}'}\left[J_1(S_1')\right] + l_1\left(\mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_1^{\mathbb{Q}'}\right] - \mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_0^{\mathbb{Q}'}\right]\right),$$

where $\mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_0^{\mathbb{Q}'}\right] = \mathbb{E}\left[\Lambda_0^{\mathbb{Q}'} \ln \Lambda_0^{\mathbb{Q}'}\right] = 0$ by (2.18). Then

$$\begin{aligned} V_1\left(\mathbb{Q}', (S_k')_{k=0}^1\right) &= \mathbb{E}_{\mathbb{Q}'}\left[J_1(S_1')\right] + l_1 \mathbb{E}_{\mathbb{Q}'}\left[\ln \Lambda_1^{\mathbb{Q}'}\right] \\ &= \sum_{\lambda\in\Omega_1} q_1'^\lambda J_1^\lambda(S_1'^\lambda) + l_1 \sum_{\lambda\in\Omega_1} q_1'^\lambda \ln \frac{q_1'^\lambda}{p_1^\lambda} \\ &= l_1 \sum_{\lambda\in\Omega_1} q_1'^\lambda \left(\ln \frac{q_1'^\lambda}{p_1^\lambda} + \frac{J_1^\lambda(S_1'^\lambda)}{l_1}\right). \end{aligned}$$

Combining Proposition 5.21 and Lemma 5.16, we have

$$\begin{aligned} V_0(\mathbb{Q}, S_0) = & \inf_{(\mathbb{Q}', (S_k')_{k=0}^1) \in \bar{\mathcal{P}}_1^0(\mathbb{Q}, S_0)} V_1\left(\mathbb{Q}', (S_k')_{k=0}^1\right) \\ = & l_1 \inf \left\{ \sum_{\lambda\in\Omega_1} w_1^\lambda \left(\ln \frac{w_1^\lambda}{p_1^\lambda} + \frac{J_1^\lambda(s_1^\lambda)}{l_1}\right) \middle| \, w_1^\lambda \in [0,1], \right. \\ & \left. s_1^\lambda \in [S_1^{b\lambda}, S_1^{a\lambda}] \, \forall \lambda \in \nu^+, \sum_{\lambda\in\nu^+} w_1^\lambda = 1, \sum_{\lambda\in\nu^+} w_1^\lambda s_1^\lambda = S_0 \right\}. \end{aligned}$$

Then the construction of J_0 in Algorithm 5.17 implies

$$V_0(\mathbb{Q}, S_0) = J_0(S_0),$$

and hence (5.47) holds true. This completes the proof. $\square$

This section ends with the proof of Theorem 5.20.

Proof of Theorem 5.20. We will first prove that $(\hat{\mathbb{Q}}, \hat{S}) \in \bar{\mathcal{P}}$, where $(\hat{\mathbb{Q}}, \hat{S})$ is constructed in Algorithm 5.19. Then we will focus on the proof of (5.43). At the end of this proof, we will show that $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$.

Firstly, we have $\hat{\mathbb{Q}}(\emptyset) = 0$ by the definition of $\hat{\mathbb{Q}}$. Let $(A_k)_{k=1}^{\infty}$ be a sequence of pairwise disjoint sets in $\mathcal{F}$. Then

$$\begin{aligned}\hat{\mathbb{Q}}\left(\bigcup_{k=1}^{\infty} A_k\right) &= \sum_{\omega \in \cup_{k=1}^{\infty} A_k} \prod_{t=0}^{T} \bar{q}_t^{\{\omega\}_t} \\ &= \sum_{k=1}^{\infty} \sum_{\omega \in A_k} \prod_{t=0}^{T} \bar{q}_t^{\{\omega\}_t} \\ &= \sum_{k=1}^{\infty} \hat{\mathbb{Q}}\left(A_k\right).\end{aligned}$$

This means that $\hat{\mathbb{Q}}$ is countably additive. We are going to prove by induction that

$$\hat{\mathbb{Q}}(\nu) = \prod_{k=0}^{t} \bar{q}_k^{\nu_k} \text{ for all } \nu \in \Omega_t \tag{5.49}$$

for each $t = 0, \ldots, T$. Observe that

$$\hat{\mathbb{Q}}(\nu) = \sum_{\omega \in \nu} \prod_{t=0}^{T} \bar{q}_t^{\{\omega\}_t} = \prod_{t=0}^{T} \bar{q}_t^{\nu_t} \text{ for all } \nu \in \Omega_T.$$

This means that (5.49) holds true for $t = T$. Fix any $i = 0, \ldots, T-1$. Suppose that (5.49) holds true for $t = i+1$. For any $\nu \in \Omega_i$, it follows from $\nu = \cup_{\lambda \in \nu^+} \lambda$ that

$$\hat{\mathbb{Q}}(\nu) = \sum_{\lambda \in \nu^+} \hat{\mathbb{Q}}(\lambda),$$

where $\hat{\mathbb{Q}}(\lambda) = \prod_{k=0}^{i+1} \bar{q}_k^{\lambda_k}$ because (5.49) holds true for $t = i+1$. Thus $\hat{\mathbb{Q}}(\nu)$ can be written as

$$\begin{aligned}\hat{\mathbb{Q}}(\nu) &= \sum_{\lambda \in \nu^+} \prod_{k=0}^{i+1} \bar{q}_k^{\lambda_k} \\ &= \sum_{\lambda \in \nu^+} \prod_{k=0}^{i} \bar{q}_k^{\lambda_k} \bar{q}_{i+1}^{\lambda} \\ &= \sum_{\lambda \in \nu^+} \prod_{k=0}^{i} \bar{q}_k^{\nu_k} \bar{q}_{i+1}^{\lambda} \\ &= \prod_{k=0}^{i} \bar{q}_k^{\nu_k} \sum_{\lambda \in \nu^+} \bar{q}_{i+1}^{\lambda}\end{aligned}$$

$$= \prod_{k=0}^{i} \bar{q}_k^{\nu_k}.$$

Therefore (5.49) holds true for $t = i$. This completes the induction step, and hence (5.49) holds true for each $t = 0, \ldots, T$. Notice that $\Omega \in \Omega_0$ and (5.49) gives $\hat{\mathbb{Q}}(\Omega) = \bar{q}_0 = 1$. The conclusion is that $\hat{\mathbb{Q}} \ll \mathbb{P}$ is a probability measure. From (5.49), the transition probabilities of $\hat{\mathbb{Q}}$ satisfy

$$\hat{q}_{i+1}^{\lambda} = \frac{\hat{\mathbb{Q}}(\lambda)}{\hat{\mathbb{Q}}(\nu)} = \frac{\prod_{k=0}^{i+1} \bar{q}_k^{\lambda_k}}{\prod_{k=0}^{i} \bar{q}_k^{\nu_k}} = \frac{\prod_{k=0}^{i} \bar{q}_k^{\nu_k} \bar{q}_{i+1}^{\lambda}}{\prod_{k=0}^{i} \bar{q}_k^{\nu_k}} = \bar{q}_{i+1}^{\lambda}$$

for all $i = 0, \ldots, T-1$, $\nu \in \Omega_i^+(\hat{\mathbb{Q}})$ and $\lambda \in \nu^+$. Moreover, from Algorithm 5.19, we have $\hat{S} = (\hat{S}_i)_{i=0}^T = (\bar{s}_i)_{i=0}^T$. Then it follows from the construction of $(\bar{q}_i, \bar{s}_i)_{i=0}^T$ that

$$\begin{gathered} S_i^b \leq \hat{S}_i \leq S_i^a \text{ for all } i = 0, \ldots, T, \\ \textstyle\sum_{\lambda \in \nu^+} \hat{q}_{i+1}^{\lambda} \hat{S}_{i+1}^{\lambda} = \hat{S}_i^{\nu} \text{ for all } i = 0, \ldots, T-1, \nu \in \Omega_i^+(\hat{\mathbb{Q}}). \end{gathered}$$

Then Lemma 5.15 implies $(\hat{\mathbb{Q}}, \hat{S}) \in \bar{\mathcal{P}}_T = \bar{\mathcal{P}}$.

Secondly, we are going to show that (5.43) holds true. It follows from the definition of $\bar{\mathcal{P}}_0$ in (5.30) and $(\hat{\mathbb{Q}}, \hat{S}) = (\hat{\mathbb{Q}}, (\hat{S}_t)_{t=0}^T) \in \bar{\mathcal{P}}$ that $(\hat{\mathbb{Q}}, \hat{S}_0) \in \bar{\mathcal{P}}_0$. Then (5.47) gives

$$\begin{aligned} V_0\left(\hat{\mathbb{Q}}, \hat{S}_0\right) &= J_0\left(\hat{S}_0\right) && (5.50) \\ &= \inf_{S_0 \in [S_0^b, S_0^a]} J_0(S_0) \\ &= \inf_{(\mathbb{Q}, S_0) \in \bar{\mathcal{P}}_0} J_0(S_0) \\ &= \inf_{(\mathbb{Q}, S_0) \in \bar{\mathcal{P}}_0} V_0(\mathbb{Q}, S_0); && (5.51) \end{aligned}$$

the last equality follows from (5.47) as well. Fix any $t = 0, \ldots, T-1$. The expectation $\mathbb{E}_{\hat{\mathbb{Q}}}[J_t(\hat{S}_t)]$ can be presented as

$$\mathbb{E}_{\hat{\mathbb{Q}}}\left[J_t(\hat{S}_t)\right] = \sum_{\nu \in \Omega_t^+(\hat{\mathbb{Q}})} \hat{\mathbb{Q}}(\nu) J_t^{\nu}(\hat{S}_t^{\nu}),$$

where

$$J_t^{\nu}(\hat{S}_t^{\nu}) = l_{t+1} \sum_{\lambda \in \nu^+} \hat{q}_{t+1}^{\lambda} \left(\ln \frac{\hat{q}_{t+1}^{\lambda}}{p_{t+1}^{\lambda}} + \frac{J_{t+1}^{\lambda}(\hat{S}_{t+1}^{\lambda})}{l_{t+1}} \right)$$

by the definition of $J_t^{\nu}(\hat{S}_t^{\nu})$ and the fact that $(\bar{q}_{t+1}^{\lambda}, \bar{s}_{t+1}^{\lambda})_{\lambda \in \nu^+} = (\hat{q}_{t+1}^{\lambda}, \hat{S}_{t+1}^{\lambda})_{\lambda \in \nu^+}$ is a solution to the problem in (5.42) with $s = \bar{s}_t^{\nu} = \hat{S}_t^{\nu}$. Then $\mathbb{E}_{\hat{\mathbb{Q}}}[J_t(\hat{S}_t)]$ can be written as

$$\mathbb{E}_{\hat{\mathbb{Q}}}\left[J_t(\hat{S}_t)\right] = l_{t+1} \sum_{\nu \in \Omega_t^+(\hat{\mathbb{Q}})} \hat{\mathbb{Q}}(\nu) \sum_{\lambda \in \nu^+} \hat{q}_{t+1}^{\lambda} \left(\ln \frac{\hat{q}_{t+1}^{\lambda}}{p_{t+1}^{\lambda}} + \frac{J_{t+1}^{\lambda}(\hat{S}_{t+1}^{\lambda})}{l_{t+1}} \right).$$

Observe from Remark 5.12 that

$$\sum_{\nu \in \Omega_t^+(\hat{\mathbb{Q}})} \hat{\mathbb{Q}}(\nu) \sum_{\lambda \in \nu^+} \hat{q}_{t+1}^{\lambda} J_{t+1}^{\lambda}(\hat{S}_{t+1}^{\lambda}) = \mathbb{E}_{\hat{\mathbb{Q}}}\left[J_{t+1}(\hat{S}_{t+1})\right].$$

Moreover, it follows from Lemma 5.14 that

$$l_{t+1} \sum_{\nu \in \Omega_t^+(\hat{\mathbb{Q}})} \hat{\mathbb{Q}}(\nu) \sum_{\lambda \in \nu^+} \hat{q}_{t+1}^{\lambda} \ln \frac{\hat{q}_{t+1}^{\lambda}}{p_{t+1}^{\lambda}} = l_{t+1} \left(\mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_{t+1}^{\hat{\mathbb{Q}}}\right] - \mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_t^{\hat{\mathbb{Q}}}\right] \right).$$

Thus

$$\mathbb{E}_{\hat{\mathbb{Q}}}\left[J_t(\hat{S}_t)\right] = \mathbb{E}_{\hat{\mathbb{Q}}}\left[J_{t+1}(\hat{S}_{t+1})\right] + l_{t+1} \left(\mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_{t+1}^{\hat{\mathbb{Q}}}\right] - \mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_t^{\hat{\mathbb{Q}}}\right] \right). \tag{5.52}$$

By adding $\sum_{k=1}^{t} l_k \left(\mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_k^{\hat{\mathbb{Q}}}\right] - \mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_{k-1}^{\hat{\mathbb{Q}}}\right] \right)$ on both sides of (5.52), we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}\left[J_t(\hat{S}_t)\right] &+ \sum_{k=1}^{t} l_k \left(\mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_k^{\hat{\mathbb{Q}}}\right] - \mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_{k-1}^{\hat{\mathbb{Q}}}\right] \right) \\ &= \mathbb{E}_{\hat{\mathbb{Q}}}\left[J_{t+1}(\hat{S}_{t+1})\right] + \sum_{k=1}^{t+1} l_k \left(\mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_k^{\hat{\mathbb{Q}}}\right] - \mathbb{E}_{\hat{\mathbb{Q}}}\left[\ln \Lambda_{k-1}^{\hat{\mathbb{Q}}}\right] \right). \end{aligned}$$

Combining this with the fact that $(\hat{\mathbb{Q}}, (\hat{S}_k)_{k=0}^{t}) \in \bar{\mathcal{P}}_t$ and $(\hat{\mathbb{Q}}, (\hat{S}_k)_{k=0}^{t+1}) \in \bar{\mathcal{P}}_{t+1}$, it follows from Proposition 5.22 that

$$V_t\left(\hat{\mathbb{Q}}, (\hat{S}_k)_{k=0}^{t}\right) = V_{t+1}\left(\hat{\mathbb{Q}}_t, (\hat{S}_k)_{k=0}^{t+1}\right).$$

Therefore, we can conclude that

$$V_T\left(\hat{\mathbb{Q}}, (\hat{S}_k)_{k=0}^{T}\right) = V_0\left(\hat{\mathbb{Q}}, \hat{S}_0\right) = \inf_{(\mathbb{Q}, S_0) \in \bar{\mathcal{P}}_0} V_0(\mathbb{Q}, S_0) \tag{5.53}$$

by (5.51). From (5.45), (5.50) and (5.46), we can present the common value in (5.53) above as follows:

$$\begin{aligned} V_T\left(\hat{\mathbb{Q}}, (\hat{S}_k)_{k=0}^{T}\right) &= H_{\mathcal{I}}\left((\hat{\mathbb{Q}}, \hat{S}); X\right), \\ V_0\left(\hat{\mathbb{Q}}, \hat{S}_0\right) &= J_0\left(\hat{S}_0\right), \end{aligned}$$

$$\inf_{(\mathbb{Q},S_0)\in\bar{\mathcal{P}}_0} V_0(\mathbb{Q},S_0) = K_{\mathcal{I}}(X).$$

Thus (5.43) holds true. It follows from

$$H_{\mathcal{I}}\left((\hat{\mathbb{Q}},\hat{S});X\right) = K_{\mathcal{I}}(X)$$

that $(\hat{\mathbb{Q}},\hat{S})$ solves the problem (5.7). Finally, Lemma 5.3 implies $(\hat{\mathbb{Q}},\hat{S})\in\mathcal{P}$. This completes the proof. □

5.3 Piecewise linear approximation

In this section, based on the assumption that (5.26) holds true, we will introduce a sequence of random functions $(\widetilde{J}_t)_{t=0}^T$ for approximating $(J_t)_{t=0}^T$ constructed in Algorithm 5.17. For any $t=0,\dots,T$ and $\nu\in\Omega_t$, we will set $\widetilde{J}_t^\nu$ to be convex and piecewise linear on $[S_t^{b\nu},S_t^{a\nu}]$, and satisfy $\widetilde{J}_t^\nu\geq J_t^\nu$. Moreover, as the number of segments of $\widetilde{J}_t^\nu$ increases, it will converge to J_t^ν uniformly; this convergence is established in Theorem 5.25. Such piecewise linear approximation allows us to compute $(\widetilde{J}_t)_{t=0}^T$ in a binary model by using the results from Section 4.3. In Section 5.5, we will discuss how to use $(\widetilde{J}_t)_{t=0}^T$ as an approximation of $(J_t)_{t=0}^T$ to compute the optimal injections, the minimal regret, and the regret indifference prices in a binary model.

Let $n\geq 2$ be an integer. For every $t=0,\dots,T-1$ and $\nu\in\Omega_t$, the bid-ask interval $[S_t^{b\nu},S_t^{a\nu}]$ is divided into $n-1$ subintervals

$$\left[s_t^{1\nu},s_t^{2\nu}\right],\ \left[s_t^{2\nu},s_t^{3\nu}\right],\ \dots,\ \left[s_t^{n-1,\nu},s_t^{n\nu}\right] \tag{5.54}$$

with equal length by taking

$$s_t^{i\nu} := S_t^{b\nu} + \tfrac{i-1}{n-1}\left(S_t^{a\nu}-S_t^{b\nu}\right) \text{ for all } i=1,\dots,n. \tag{5.55}$$

Observe that

$$\left[s_t^{1\nu},s_t^{n\nu}\right] = \left[S_t^{b\nu},S_t^{a\nu}\right].$$

Consider the following two cases. In the case when $S_t^{b\nu}<S_t^{a\nu}$, we have

$$s_t^{1\nu}<s_t^{2\nu}<\cdots<s_t^{n\nu}. \tag{5.56}$$

In the case when $S_t^{b\nu}=S_t^{a\nu}$ (i.e. the transaction costs are zero at time t on the node ν), the interval $\left[s_t^{1\nu},s_t^{n\nu}\right]$ is a singleton set, and this leads to

$$s_t^{1\nu}=s_t^{2\nu}=\cdots=s_t^{n\nu}.$$

We will set $\widetilde{J}_t^\nu$ to be affine on each of the intervals in (5.54). Then $\widetilde{J}_t^\nu$ will be piecewise linear with $n-1$ segments on $[s_t^{1\nu}, s_t^{n\nu}]$.

The maximum distance between $s_t^{i\nu}$ and $s_t^{i+1,\nu}$ among all $t = 0, \ldots, T-1$, $\nu \in \Omega_t$ and $i = 1, \ldots, n-1$ is defined as

$$\begin{aligned}\delta(n) :=& \max\left\{ s_t^{i+1,\nu} - s_t^{i\nu} \,\middle|\, t = 0, \ldots, T-1, \nu \in \Omega_t, i = 1, \ldots, n-1 \right\} \\ =& \tfrac{1}{n-1} \max\left\{ S_t^{a\nu} - S_t^{b\nu} \,\middle|\, t = 0, \ldots, T-1, \nu \in \Omega_t \right\}.\end{aligned} \tag{5.57}$$

If there are no transaction costs at each time $t = 0, \ldots, T-1$, then $\delta(n) = 0$. Notice that

$$\lim_{n\to\infty} \delta(n) = 0. \tag{5.58}$$

Remark 5.23. We provide a concrete construction for the intervals (5.54). However, the main results (Proposition 5.24 and Theorem 5.25) established in this section does not rely on the fact that the intervals are of the same width. As long as the end points of these intervals satisfy (5.56) when $S_t^{b\nu} < S_t^{a\nu}$ for all $t = 0, \ldots, T-1$ and $\nu \in \Omega_t$, the main results still hold true.

To approximate $(J_t)_{t=0}^T$, we are going to define $\mathcal{F}_t$-measurable random functions $\widetilde{J}_t$ and $\widetilde{J}_t^*$ recursively for each $t = T, \ldots, 0$; the function $\widetilde{J}_t^*$ is an auxiliary function used to construct $\widetilde{J}_t$. For any $\nu \in \Omega_T$ and $s \in \mathbb{R}$, we define

$$\widetilde{J}_T^\nu(s) \equiv \widetilde{J}_T^{*\nu}(s) := J_T^\nu(s) = \begin{cases} (1, s) \cdot X^\nu & \text{if } s \in \left[S_T^{b\nu}, S_T^{a\nu}\right], \\ \infty & \text{otherwise.} \end{cases} \tag{5.59}$$

Let $t = T-1, \ldots, 0$ and $\nu \in \Omega_t$. For any $s \in \mathbb{R}$, we define

$$\begin{aligned}\widetilde{J}_t^{*\nu}(s) := l_{t+1} \inf_{(q_{t+1}^\lambda, s_{t+1}^\lambda)_{\lambda\in\nu^+}} &\left\{ \sum_{\lambda\in\nu^+} q_{t+1}^\lambda \left(\ln \frac{q_{t+1}^\lambda}{p_{t+1}^\lambda} + \frac{\widetilde{J}_{t+1}^\lambda(s_{t+1}^\lambda)}{l_{t+1}} \right) \,\middle|\, q_{t+1}^\lambda \in [0,1], \right. \\ &\left. s_{t+1}^\lambda \in \left[S_{t+1}^{b\lambda}, S_{t+1}^{a\lambda}\right] \ \forall \lambda \in \nu^+, \sum_{\lambda\in\nu^+} q_{t+1}^\lambda = 1, \sum_{\lambda\in\nu^+} q_{t+1}^\lambda s_{t+1}^\lambda = s \right\}.\end{aligned} \tag{5.60}$$

Observe from (5.42) that if $J_{t+1}^\lambda \leq \widetilde{J}_{t+1}^\lambda$ for all $\lambda \in \nu^+$, then $J_t^\nu \leq \widetilde{J}_t^{*\nu}$ on $[S_t^{b\nu}, S_t^{a\nu}]$. Moreover, we have $\widetilde{J}_{T-1}^* = J_{T-1}$. It turns out that $\widetilde{J}_t^{*\nu}$ is real-valued, convex, and continuous on $[S_t^{b\nu}, S_t^{a\nu}]$; see Proposition 5.24.1. We shall define $\widetilde{J}_t^\nu$ by considering the following two cases. In the case when $s_t^{1\nu} = s_t^{n\nu}$, the function $\widetilde{J}_t^\nu$ is defined as

$$\widetilde{J}_t^\nu := \begin{cases} \widetilde{J}_t^{*\nu} & \text{on } [s_t^{1\nu}, s_t^{n\nu}] = [S_t^{b\nu}, S_t^{a\nu}] = \{s_t^{1\nu}\}, \\ \infty & \text{on } \mathbb{R} \backslash \{s_t^{1\nu}\}. \end{cases}$$

In the case when $s_t^{1\nu} < s_t^{n\nu}$, the sequence $(s_t^{i\nu})_{i=1}^n$ satisfies (5.56). By connecting $(s_t^{i\nu}, \widetilde{J}_t^{*\nu}(s_t^{i\nu}))$ and $(s_t^{i+1,\nu}, \widetilde{J}_t^{*\nu}(s_t^{i+1,\nu}))$ for each $i = 1, \dots, n-1$, we shall set $\widetilde{J}_t^{\nu}$ to be continuous and piecewise linear on $[s_t^{1\nu}, s_t^{n\nu}]$; an example of $\widetilde{J}_t^{\nu}$ on $[s_t^{1\nu}, s_t^{n\nu}]$ with $\widetilde{J}_t^{*\nu}$ given is provided in Figure 5.1. Let

$$\widetilde{J}_t^{\nu}(s) := \widetilde{J}_t^{*\nu}(s) \text{ for all } s = s_t^{i\nu}, \dots, s_t^{n\nu}. \tag{5.61}$$

Moreover, for any $i = 1, \dots, n-1$, let

$$\widetilde{J}_t^{\nu}(s) := \widetilde{m}_t^{i\nu} s + \widetilde{J}_t^{\nu *}\left(s_t^{i\nu}\right) - \widetilde{m}_t^{i\nu} s_t^{i\nu} \text{ for all } s \in (s_t^{i\nu}, s_t^{i+1,\nu}) \tag{5.62}$$

where

$$\widetilde{m}_t^{i\nu} := \frac{\widetilde{J}_t^{*\nu}(s_t^{i+1,\nu}) - \widetilde{J}_t^{*\nu}\left(s_t^{i\nu}\right)}{s_t^{i+1,\nu} - s_t^{i\nu}}.$$

As long as $\widetilde{J}_t^{*\nu}$ is real-valued, continuous, and convex on $[s_t^{1\nu}, s_t^{n\nu}]$, the function $\widetilde{J}_t^{\nu}$ is real-valued, continuous, piecewise linear, convex, and it satisfies $\widetilde{J}_t^{\nu} \geq \widetilde{J}_t^{*\nu}$ on $[s_t^{1\nu}, s_t^{n\nu}]$; see Lemma A.10. Finally, let

$$\widetilde{J}_t^{\nu} := \infty \text{ on } \mathbb{R}\setminus\left[s_t^{1\nu}, s_t^{n\nu}\right]. \tag{5.63}$$

This completes the definitions of $(\widetilde{J}_t^*)_{t=0}^T$ and $(\widetilde{J}_t)_{t=0}^T$. Notice that for any $t = 0, \dots, T-1$ and $\nu \in \Omega_t$, the function $\widetilde{J}_t^{\nu}$ relies on $\widetilde{J}_t^{*\nu}(s_t^{1\nu}), \dots, \widetilde{J}_t^{*\nu}(s_t^{n\nu})$ only. It turns out that $(\widetilde{J}_t)_{t=0}^T$ can be used to approximate $(J_t)_{t=0}^T$ as long as $\delta(n)$ is close to 0. The relevant convergence result is provided in Theorem 5.25.

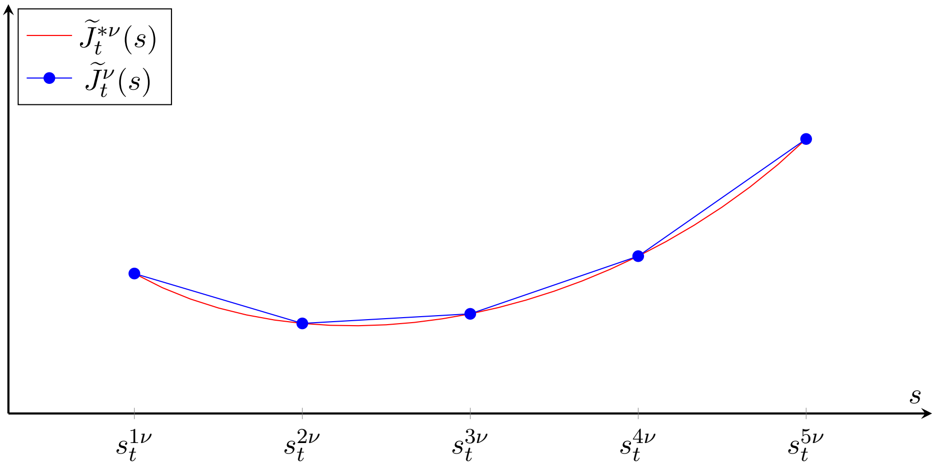


Figure 5.1: Picture of $\widetilde{J}_t^{\nu}$ with $\widetilde{J}_t^{*\nu}$ given, where $t < T$, $\nu \in \Omega_t$ and $n = 5$

Proposition 5.24 below provides a number of properties of $\widetilde{J}_t^{*\nu}$ and $\widetilde{J}_t^{\nu}$ on $[S_t^{b\nu}, S_t^{a\nu}]$ for each $t = 0, \dots, T$ and $\nu \in \Omega_t$. Moreover, it shows that the minimisation problem in (5.60) with $s \in [S_t^{b\nu}, S_t^{a\nu}]$ admits a solution. Some results from Sections 4.1-4.2 will be used in the proof of the following result.

Proposition 5.24. *Let $t = 0, \dots, T$. The following two claims hold true.*

1. *For any $\nu \in \Omega_t$, on $[S_t^{b\nu}, S_t^{a\nu}]$, the functions $s \mapsto \widetilde{J}_t^{*\nu}(s)$ and $s \mapsto \widetilde{J}_t^{\nu}(s)$ satisfy the following properties.*
   - *The function $\widetilde{J}_t^{*\nu}$ is real-valued, continuous, and convex.*
   - *The function $\widetilde{J}_t^{\nu}$ is real-valued, continuous, piecewise linear, and convex. If $t \leq T-1$, then it is affine on $[s_k^{i\nu}, s_k^{i+1,\nu}]$ for each $i = 1, \dots, n-1$.*
   - *We have $J_t^{\nu} \leq \widetilde{J}_t^{*\nu} \leq \widetilde{J}_t^{\nu}$.*
2. *For any $\nu \in \Omega_t$, if $t \leq T-1$, then there exists a solution to the minimisation problem in (5.60) for all $s \in [S_t^{b\nu}, S_t^{a\nu}]$.*

Proof. Firstly, we are going to show that the first claim holds true for each $t = T, \dots, 0$ by backward induction. For any $\nu \in \Omega_T$, it follows from (5.59) that $\widetilde{J}_T^{\nu} = \widetilde{J}_T^{*\nu} = J_T^{\nu}$ is affine on $[S_T^{b\nu}, S_T^{a\nu}]$. Thus, the first claim holds true for $t = T$. Let $k = 0, \dots, T-1$, and suppose now that the first claim holds true for $t = k+1$. Fix any $\nu \in \Omega_k$. Since the first claim holds true for $t = k+1$, for any $\lambda \in \nu^+$, the function $\widetilde{J}_{k+1}^{\lambda}$ is real-valued, continuous and convex on $[S_{k+1}^{b\lambda}, S_{k+1}^{a\lambda}]$. Moreover, it satisfies

$$J_{k+1}^{\lambda} \leq \widetilde{J}_{k+1}^{\lambda} \text{ on } \left[S_{k+1}^{b\lambda}, S_{k+1}^{a\lambda}\right] \tag{5.64}$$

From (5.63), we have

$$\widetilde{J}_{k+1}^{\lambda} = \infty \text{ on } \mathbb{R} \backslash \left[s_{k+1}^{1\lambda}, s_{k+1}^{n\lambda}\right] = \mathbb{R} \backslash \left[S_{k+1}^{b\lambda}, S_{k+1}^{a\lambda}\right].$$

Combining this with the fact that $\widetilde{J}_{k+1}^{\lambda}$ on $[S_{k+1}^{b\lambda}, S_{k+1}^{a\lambda}]$ is real-valued, we have

$$\operatorname{dom} \widetilde{J}_{k+1}^{\lambda} = \left[S_{k+1}^{b\lambda}, S_{k+1}^{a\lambda}\right]. \tag{5.65}$$

Now, we are going to prove that first claim hold true for $t = k$. For any $s \in [S_k^{b\nu}, S_k^{a\nu}]$, the condition (5.26) gives

$$s \in \left[\min_{\lambda \in \nu^+} S_{k+1}^{b\lambda}, \max_{\lambda \in \nu^+} S_{k+1}^{a\lambda}\right] = \operatorname{co}\left(\bigcup_{\lambda \in \nu^+} \left[S_{k+1}^{b\lambda}, S_{k+1}^{a\lambda}\right]\right),$$

where $\operatorname{co}(A)$ is the convex hull of a given set A. Combining this with (5.65), it follows that

$$s \in \operatorname{co}\left(\bigcup_{\lambda \in \nu^+} \operatorname{dom} \widetilde{J}_{k+1}^{\lambda}\right).$$

Then the value $\widetilde{J}_k^{*\nu}(s)$ is finite by (4.5). Thus $\widetilde{J}_k^{*\nu}$ is real-valued on $[S_k^{b\nu}, S_k^{a\nu}]$. Moreover, from Theorems 4.3 and 4.13, the function $\widetilde{J}_k^{*\nu}$ is convex and continuous on $[S_k^{b\nu}, S_k^{a\nu}]$. Since (5.64) holds true for all $\lambda \in \nu^+$, it follows from the definitions of J_k^ν and $\widetilde{J}_k^{*\nu}$ (see (5.42) and (5.60)) that $J_k^\nu \leq \widetilde{J}_k^{*\nu}$ on $[S_k^{b\nu}, S_k^{a\nu}]$. From (5.61)-(5.62), the construction of $\widetilde{J}_k^\nu$ is based on $\widetilde{J}_k^{*\nu}$. Since $\widetilde{J}_k^{*\nu}$ is real-valued, continuous, and convex on $[S_k^{b\nu}, S_k^{a\nu}]$, the function $\widetilde{J}_k^\nu$ is real-valued, continuous, piecewise linear, convex, and it satisfies $\widetilde{J}_k^\nu \geq \widetilde{J}_k^{*\nu}$ on $[S_k^{b\nu}, S_k^{a\nu}]$; see Lemma A.10. Clearly, the function $\widetilde{J}_k^\nu$ is affine on $[s_k^{i\nu}, s_k^{i+1,\nu}]$ for each $i = 1, \ldots, n-1$. We can conclude that all conditions in the first claim hold true for $t = k$. This completes the induction step, and hence the first claim holds true for all $t = 0, \ldots, T$. The second claim follows from Theorem 4.13. □

The theorem below shows that $(\widetilde{J}_t)_{t=0}^T$ can be used to approximate $(J_t)_{t=0}^T$ when $\delta(n)$ defined in (5.57) is close to 0. More precisely, it shows that, for any $t = 0, \ldots, T$ and $\nu \in \Omega_t$, the piecewise linear function $\widetilde{J}_t^\nu$ converges to J_t^ν uniformly as $n \to \infty$.

Theorem 5.25. *Under the assumption that* (5.26) *holds true, for any* $t = 0, \ldots, T$*, there exists* $B_t \in \mathcal{L}_{t+}$*, which is independent of* n*, such that*

$$\left|\widetilde{J}_t^\nu(s) - J_t^\nu(s)\right| \leq B_t^\nu \delta(n) \text{ for all } \nu \in \Omega_t, s \in \left[S_t^{b\nu}, S_t^{a\nu}\right]. \tag{5.66}$$

Proof. We are going to prove this theorem by backward induction. At time $t = T$, let $B_T := 0$ which is independent of n. Since $\widetilde{J}_T = J_T$ by (5.59), the condition (5.66) holds true for $t = T$. Let $k = 0, \ldots, T-1$. Suppose that there exists $B_{k+1} \in (\mathcal{L}_{k+1})_+$, which is independent of n, such that (5.66) holds true for $t = k+1$. From Proposition 5.18.3, there exists $A_k \in \mathcal{L}_{k+}$, which is independent of n, such that

$$|J_k^\nu(x_1) - J_k^\nu(x_2)| \leq A_k^\nu |x_1 - x_2| \text{ for all } \nu \in \Omega_k, x_1, x_2 \in \left[S_k^{b\nu}, S_k^{a\nu}\right]. \tag{5.67}$$

Then we define $B_k \in \mathcal{L}_{k+}$ as

$$B_k^\nu := \sum_{\lambda \in \nu^+} B_{k+1}^\lambda + A_k^\nu \geq 0 \text{ for all } \nu \in \Omega_k.$$

Notice that B_k is independent of n. We are going to prove (5.66) holds true for $t = k$. Fix any $\nu \in \Omega_k$ and $s \in [S_k^{b\nu}, S_k^{a\nu}] = [s_k^{1\nu}, s_k^{n\nu}]$. Notice that

$$s \in \left[s_k^{j\nu}, s_k^{j+1,\nu}\right] \text{ for some } j = 1, \ldots, n-1.$$

By Proposition 5.24.1, the function $\widetilde{J}_k^\nu$ is affine on $[s_k^{j\nu}, s_k^{j+1,\nu}]$. Then by letting

$s' \in \{s_k^{j\nu}, s_k^{j+1,\nu}\}$ such that

$$\widetilde{J}_k^{\nu}(s') = \max\left\{\widetilde{J}_k^{\nu}(s_k^{j\nu}), \widetilde{J}_k^{\nu}(s_k^{j+1,\nu})\right\},$$

we have

$$\widetilde{J}_k^{\nu}(s) \leq \widetilde{J}_k^{\nu}(s').$$

Moreover, by subtracting $J_k^{\nu}(s)$ on both sides, it follows that

$$\widetilde{J}_k^{\nu}(s) - J_k^{\nu}(s) \leq \widetilde{J}_k^{\nu}(s') - J_k^{\nu}(s).$$

Observe from Proposition 5.24.1 that $\widetilde{J}_k^{\nu}(s) - J_k^{\nu}(s) \geq 0$. Then

$$\begin{aligned} \left|\widetilde{J}_k^{\nu}(s) - J_k^{\nu}(s)\right| &\leq \left|\widetilde{J}_k^{\nu}(s') - J_k^{\nu}(s)\right| \\ &\leq \left|\widetilde{J}_k^{\nu}(s') - J_k^{\nu}(s')\right| + \left|J_k^{\nu}(s') - J_k^{\nu}(s)\right|. \end{aligned} \tag{5.68}$$

Since $\widetilde{J}_k^{\nu}(s') = \widetilde{J}_k^{*\nu}(s')$, the value $\left|\widetilde{J}_k^{\nu}(s') - J_k^{\nu}(s')\right|$ in (5.68) can be written as

$$\left|\widetilde{J}_k^{\nu}(s') - J_k^{\nu}(s')\right| = \left|\widetilde{J}_k^{*\nu}(s') - J_k^{\nu}(s')\right|.$$

In addition, from Lemma A.6 together with the definitions of $\widetilde{J}_k^{*\nu}(s')$ and $J_k^{\nu}(s')$, it follows that

$$\begin{aligned} &\left|\widetilde{J}_k^{\nu}(s') - J_k^{\nu}(s')\right| \\ &\leq \sup_{(q_{k+1}^{\lambda}, s_{k+1}^{\lambda})_{\lambda\in\nu^+}} \left\{ \left| \sum_{\lambda\in\nu^+} q_{k+1}^{\lambda}\left(\widetilde{J}_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right) - J_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)\right)\right| \,\middle|\, q_{k+1}^{\lambda} \in [0,1], \right. \\ &\qquad \left. s_{k+1}^{\lambda} \in \left[S_{k+1}^{b\lambda}, S_{k+1}^{a\lambda}\right] \ \forall\lambda\in\nu^+, \sum_{\lambda\in\nu^+} q_{k+1}^{\lambda} = 1, \sum_{\lambda\in\nu^+} q_{k+1}^{\lambda} s_{k+1}^{\lambda} = s' \right\}. \\ &\leq \sup_{(q_{k+1}^{\lambda}, s_{k+1}^{\lambda})_{\lambda\in\nu^+}} \left\{ \sum_{\lambda\in\nu^+} q_{k+1}^{\lambda}\left|\widetilde{J}_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right) - J_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)\right| \,\middle|\, q_{k+1}^{\lambda} \in [0,1], \right. \\ &\qquad \left. s_{k+1}^{\lambda} \in \left[S_{k+1}^{b\lambda}, S_{k+1}^{a\lambda}\right] \ \forall\lambda\in\nu^+, \sum_{\lambda\in\nu^+} q_{k+1}^{\lambda} = 1, \sum_{\lambda\in\nu^+} q_{k+1}^{\lambda} s_{k+1}^{\lambda} = s' \right\}. \end{aligned} \tag{5.69}$$

In (5.69), we have $q_{k+1}^{\lambda} \in [0,1]$ and

$$\left|\widetilde{J}_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right) - J_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)\right| \geq 0$$

This implies that

$$q_{k+1}^{\lambda}\left|\widetilde{J}_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)-J_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)\right|\leq\left|\widetilde{J}_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)-J_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)\right|.$$

Thus

$$\begin{aligned}
&\left|\widetilde{J}_{k}^{\nu}\left(s'\right)-J_{k}^{\nu}\left(s'\right)\right|\\
&\leq\sup_{(q_{k+1}^{\lambda},s_{k+1}^{\lambda})_{\lambda\in\nu^{+}}}\left\{\sum_{\lambda\in\nu^{+}}\left|\widetilde{J}_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)-J_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)\right|\,\middle|\,q_{k+1}^{\lambda}\in[0,1],\right.\\
&\qquad\left.s_{k+1}^{\lambda}\in\left[S_{k+1}^{b\lambda},S_{k+1}^{a\lambda}\right]\ \forall\lambda\in\nu^{+},\ \sum_{\lambda\in\nu^{+}}q_{k+1}^{\lambda}=1,\ \sum_{\lambda\in\nu^{+}}q_{k+1}^{\lambda}s_{k+1}^{\lambda}=s'\right\}\\
&\leq\sup\left\{\sum_{\lambda\in\nu^{+}}\left|\widetilde{J}_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)-J_{k+1}^{\lambda}\left(s_{k+1}^{\lambda}\right)\right|\,\middle|\,s_{k+1}^{\lambda}\in\left[S_{k+1}^{b\lambda},S_{k+1}^{a\lambda}\right]\ \forall\lambda\in\nu^{+}\right\}.
\end{aligned}$$

Combining this with the fact that (5.66) holds true for $t=k+1$, we have

$$\left|\widetilde{J}_{k}^{\nu}(s')-J_{k}^{\nu}(s')\right|\leq\sum_{\lambda\in\nu^{+}}B_{k+1}^{\lambda}\delta(n). \tag{5.70}$$

Moreover, by (5.67), the quantity $\left|J_{k}^{\nu}\left(s'\right)-J_{k}^{\nu}(s)\right|$ in (5.68) satisfies

$$\left|J_{k}^{\nu}\left(s'\right)-J_{k}^{\nu}(s)\right|\leq A_{k}^{\nu}\left|s'-s\right|\leq A_{k}^{\nu}\delta(n). \tag{5.71}$$

Therefore, we can conclude from (5.68), (5.70) and (5.71) that

$$\left|\widetilde{J}_{k}^{\nu}(s)-J_{k}^{\nu}(s)\right|\leq\sum_{\lambda\in\nu^{+}}B_{k+1}^{\lambda}\delta(n)+A_{k}^{\nu}\delta(n)=B_{k}^{\nu}\delta(n)$$

which means that (5.66) holds true for $t=k$. This completes the induction step, and hence the result follows. □

5.4 Approximation error

In Section 5.3, a sequence of random functions $(\widetilde{J}_t)_{t=0}^T$ was defined to approximate $(J_t)_{t=0}^T$ constructed in Algorithm 5.17. For any $t=0,\ldots,T$, the function J_t is dominated by $\widetilde{J}_t$; see Proposition 5.24.1. The objective of this section is to find an upper bound of the approximation error for approximating $(J_t)_{t=0}^T$ by using $(\widetilde{J}_t)_{t=0}^T$. To achieve this, under (5.26), we will construct a sequence of random functions $(\check{J}_t)_{t=0}^T$ such that $\check{J}_t\leq J_t$ for all $t=0,\ldots,T$. Then $\widetilde{J}_t-\check{J}_t$ is an upper bound for the approximation error $\widetilde{J}_t-J_t$. Moreover, for any $\nu\in\Omega_t$, the function $\check{J}_t^\nu$ will be convex and piecewise linear on its effective domain. By using the result from Section 4.3, we can compute $(\check{J}_t)_{t=0}^T$ in binary models, and numerical examples will be provided in Section 5.5.1.

Let $n \geq 2$ be an integer. For any $t = 0, \ldots, T-1$ and $\nu \in \Omega_t$, by using the same method in Section 5.3, we first divide the bid-ask interval $[S_t^{b\nu}, S_t^{a\nu}]$ into $n-1$ intervals

$$[s_t^{1\nu}, s_t^{2\nu}],\ [s_t^{2\nu}, s_t^{3\nu}],\ \ldots,\ [s_t^{n-1,\nu}, s_t^{n\nu}] \tag{5.72}$$

with equal length; see (5.55) for the concrete definitions of $s_t^{1\nu}, \ldots, s_t^{n\nu}$. Then we define

$$\left[s_t^{0\nu}, s_t^{n+1,\nu}\right] := \left[\min_{\lambda \in \nu^+} S_{t+1}^{b\lambda}, \max_{\lambda \in \nu^+} S_{t+1}^{a\lambda}\right]. \tag{5.73}$$

Observe that $\left[s_t^{1\nu}, s_t^{n\nu}\right] \subset \left[s_t^{0\nu}, s_t^{n+1,\nu}\right]$ because $s_t^{1\nu} > s_t^{0\nu}$ and $s_t^{n\nu} < s_t^{n+1,\nu}$; see (5.26). If $S_t^{b\nu} = S_t^{a\nu}$ which means there are no transaction costs at time t on the node ν, then $[s_t^{1\nu}, s_t^{n\nu}] = [S_t^{b\nu}, S_t^{a\nu}]$ is a singleton set and the sequence $(s_t^{i\nu})_{i=0}^{n+1}$ satisfies

$$s_t^{0\nu} < s_t^{1\nu} = s_t^{2\nu} = \cdots = s_t^{n\nu} < s_t^{n+1,\nu}.$$

However, if $S_t^{b\nu} < S_t^{a\nu}$, then sequence of numbers $(s_t^{i\nu})_{i=0}^{n+1}$ is increasing, in other words,

$$s_t^{0\nu} < s_t^{1\nu} < s_t^{2\nu} < \cdots < s_t^{n\nu} < s_t^{n+1,\nu}. \tag{5.74}$$

Remark 5.26. The main result of this section is Proposition 5.33. This result will not rely on the fact that the intervals in (5.72) are of the same width. It only requires that (5.74) holds true in the case when $S_t^{b\nu} < S_t^{a\nu}$ for every $t = 0, \ldots, T-1$ and $\nu \in \Omega_t$.

We will define $\mathcal{F}_t$-measurable random functions $\check{J}_t, \check{J}_t^*$ recursively for each $t = T, \ldots, 0$; the function $\check{J}_t^*$ is an auxiliary function used to construct $\check{J}_t$. For any $\nu \in \Omega_T$ and $s \in \mathbb{R}$, let

$$\check{J}_T^\nu(s) \equiv \check{J}_T^{*\nu}(s) := J_T^\nu(s) = \begin{cases} (1, s) \cdot X^\nu & \text{if } s \in \left[S_T^{b\nu}, S_T^{a\nu}\right], \\ \infty & \text{otherwise.} \end{cases} \tag{5.75}$$

Let $t = T-1, \ldots, 0$ and $\nu \in \Omega_t$. For all $s \in \mathbb{R}$, we define

$$\begin{aligned} \check{J}_t^{*\nu}(s) := l_{t+1} \inf_{(q_{t+1}^\lambda, s_{t+1}^\lambda)_{\lambda \in \nu^+}} \Bigg\{ \sum_{\lambda \in \nu^+} q_{t+1}^\lambda \left(\ln \frac{q_{t+1}^\lambda}{p_{t+1}^\lambda} + \frac{\check{J}_{t+1}^\lambda(s_{t+1}^\lambda)}{l_{t+1}} \right) \Bigg| \, q_{t+1}^\lambda \in [0,1], \\ s_{t+1}^\lambda \in \left[S_{t+1}^{b\lambda}, S_{t+1}^{a\lambda}\right] \ \forall \lambda \in \nu^+, \sum_{\lambda \in \nu^+} q_{t+1}^\lambda = 1, \sum_{\lambda \in \nu^+} q_{t+1}^\lambda s_{t+1}^\lambda = s \Bigg\} \end{aligned} \tag{5.76}$$

(cf. (5.42)). Lemma 5.27 below shows that if $\check{J}_{t+1}^\lambda$ is convex and continuous on its effective domain $\operatorname{dom} \check{J}^{\lambda}_{t+1} = [S^{b\lambda}_{t+1}, S^{a\lambda}_{t+1}]$ for all $\lambda \in \nu^+$, then $\check{J}^{*\nu}_t$ defined in (5.76) is also convex and continuous on its effective domain. Moreover, if $\check{J}^{\lambda}_{t+1}$ is dominated by J^{λ}_{t+1} for all $\lambda \in \nu^+$ then $\check{J}^{*\nu}_t$ is dominated by J^{ν}_t on the interval $[S^{b\nu}_t, S^{a\nu}_t]$. Observe from (5.75) that at time $t = T-1$, all assumptions on $(\check{J}^{\lambda}_{t+1})_{\lambda\in\nu^+}$ in Lemma 5.27 are satisfied.

Lemma 5.27. *Suppose that $\check{J}^{\lambda}_{t+1} : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is a convex function that is continuous on $\operatorname{dom} \check{J}^{\lambda}_{t+1} = [S^{b\lambda}_{t+1}, S^{a\lambda}_{t+1}]$ for all $\lambda \in \nu^+$. Then $\check{J}^{*\nu}_t$ defined in (5.76) is an $\mathbb{R} \cup \{\infty\}$-valued convex function on $\mathbb{R}$ that is continuous on $\operatorname{dom} \check{J}^{*\nu}_t = [s^{0\nu}_t, s^{n+1,\nu}_t]$. In addition, if $\check{J}^{\lambda}_{t+1} \leq J^{\lambda}_{t+1}$ for all $\lambda \in \nu^+$, then $\check{J}^{*\nu}_t \leq J^{\nu}_t$ on $[S^{b\nu}_t, S^{a\nu}_t]$.*

The proof of Lemma 5.27 above will be provided at the end of this section. Some results established in Sections 4.1-4.2 will be used in the proof of this lemma. Suppose now that $\check{J}^{*\nu}_t$ is real-valued, continuous, and convex on $[s^{0\nu}_t, s^{n+1,\nu}_t]$; this holds true at $t = T-1$ by Lemma 5.27 and the definition of $\check{J}_T$ in (5.75). Based on such $\check{J}^{*\nu}_t$, we will define $\check{J}^{\nu}_t : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ to be a convex function that is continuous, piecewise linear and dominated by $\check{J}^{*\nu}_t$ on $\operatorname{dom} \check{J}^{\nu}_t = [S^{b\nu}_t, S^{a\nu}_t]$. Consider the following two cases for $[s^{1\nu}_t, s^{n\nu}_t]$.

In the case when $s^{1\nu}_t = s^{n\nu}_t$, let

$$\check{J}^{\nu}_t := \begin{cases} \check{J}^{*\nu}_t & \text{on } \left[s^{1\nu}_t, s^{n\nu}_t\right] = \left[S^{b\nu}_t, S^{a\nu}_t\right] = \left\{S^{b\nu}_t\right\}, \\ \infty & \text{on } \mathbb{R}\backslash\left\{S^{b\nu}_t\right\}. \end{cases} \tag{5.77}$$

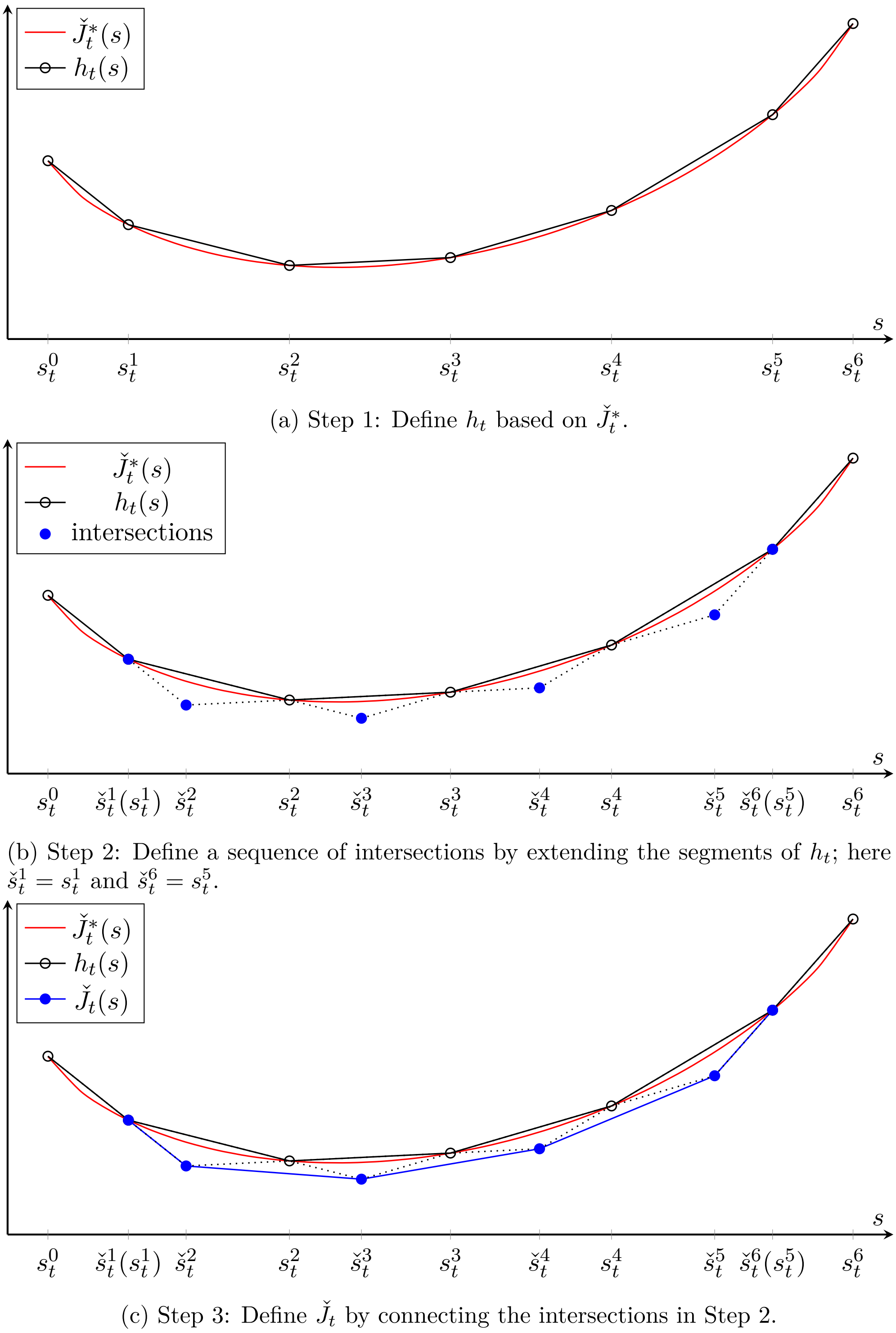
Notice that all desired properties for $\check{J}^{\nu}_t$ mentioned in the comments preceding (5.77) are satisfied.

In the case when $s^{1\nu}_t < s^{n\nu}_t$, the definition of $\check{J}^{\nu}_t$ is similar to but more complicated than $\tilde{J}^{\nu}_t$ defined in Section 5.3. We will define $\check{J}^{\nu}_t$ in three steps; Figure 5.2 provides an example to demonstrate the procedure of defining $\check{J}^{\nu}_t$. The first step is to construct a convex and piecewise linear function h^{ν}_t with $n+1$ segments on $[s^{0\nu}_t, s^{n+1,\nu}_t]$ by connecting $(s^{i-1,\nu}_t, \check{J}^{*\nu}_t(s^{i-1,\nu}_t))$ and $(s^{i\nu}_t, \check{J}^{*\nu}_t(s^{i\nu}_t))$ for each $i = 1, \ldots, n+1$; see Figure 5.2(a). The second step is to select a sequence of intersections $(\check{s}^i_t, \check{y}^i_t)^{n+1}_{i=1}$ by extending the segments of h^{ν}_t; see Figure 5.2(b). The third step is to define $\check{J}^{\nu}_t$ by connecting these intersections; see Figure 5.2(c). In the following detailed construction of $\check{J}^{\nu}_t$ (pp. 175-180), we shall always suppress ν for simplicity.

For any $i = 1, \ldots, n+1$, we define h^i_t as the affine function such that

$$h^i_t\left(s^{i-1}_t\right) = \check{J}^*_t\left(s^{i-1}_t\right), \tag{5.78}$$

$$h^i_t\left(s^{i}_t\right) = \check{J}^*_t\left(s^{i}_t\right), \tag{5.79}$$



(a) Step 1: Define h_t based on $\check{J}_t^*$.

(b) Step 2: Define a sequence of intersections by extending the segments of h_t; here $\check{s}_t^1 = s_t^1$ and $\check{s}_t^6 = s_t^5$.

(c) Step 3: Define $\check{J}_t$ by connecting the intersections in Step 2.

Figure 5.2: The procedure of defining $\check{J}_t^\nu$ based on $\check{J}_t^{*\nu}$, where $t < T$, $\nu \in \Omega_t$ and $n = 5$ (ν is suppressed in this figure)

in other words,

$$h_t^i(s) = m_t^i s + b_t^i \text{ for all } s \in \mathbb{R}$$

where

$$m_t^i = \frac{\check{J}_t^*\left(s_t^i\right) - \check{J}_t^*\left(s_t^{i-1}\right)}{s_t^i - s_t^{i-1}}, \qquad b_t^i = \check{J}_t^*\left(s_t^{i-1}\right) - m_t^i s_t^{i-1}.$$

The affine function h_t^i corresponds to the straightline connecting the two points $(s_t^{i-1}, \check{J}_t^*(s_t^{i-1}))$ and $(s_t^i, \check{J}_t^*(s_t^i))$. Notice that

$$h_t^i\left(s_t^i\right) = \check{J}_t^*\left(s_t^i\right) = h_t^{i+1}\left(s_t^i\right) \text{ for every } i = 1, \ldots, n. \tag{5.80}$$

By connecting $(s_t^{i-1}, \check{J}_t^*(s_t^{i-1}))$ and $(s_t^i, \check{J}_t^*(s_t^i))$ for each $i = 1, \ldots, n+1$, the real-valued continuous piecewise linear function h_t on $[s_t^0, s_t^{n+1}]$ is defined as

$$h_t := h_t^i \text{ on } \left[s_t^{i-1}, s_t^i\right) \text{ for all } i = 1, \ldots, n, \tag{5.81}$$

$$h_t := h_t^{n+1} \text{ on } \left[s_t^n, s_t^{n+1}\right]; \tag{5.82}$$

see Figure 5.2(a). Then h_t satisfies

$$h_t = h_t^i \text{ on } \left[s_t^{i-1}, s_t^i\right] \text{ for all } i = 1, \ldots, n+1. \tag{5.83}$$

From Lemma A.10, the slopes of the affine functions $h_t^1, \ldots, h_t^{n+1}$ satisfy

$$m_t^1 \leq \cdots \leq m_t^{n+1}, \tag{5.84}$$

and h_t is convex and satisfies $h_t \geq \check{J}_t^*$ on $[s_t^0, s_t^{n+1}]$.

Remark 5.28. For any $i = 1, \ldots, n$, if $m_t^i = m_t^{i+1}$ then $b_t^i = b_t^{i+1}$ which means $h_t^i = h_t^{i+1}$. Indeed, we have

$$b_t^i = m_t^i s_t^i + b_t^i - m_t^i s_t^i = h_t^i(s_t^i) - m_t^i s_t^i.$$

Combining this with $h_t^i(s_t^i) = h_t^{i+1}(s_t^i)$ (see (5.80)) and $m_t^i = m_t^{i+1}$, it follows that

$$b_t^i = h_t^{i+1}(s_t^i) - m_t^{i+1} s_t^i = b_t^{i+1}.$$

Our next objective is to select a sequence of intersections $(\check{s}_t^i, \check{y}_t^i)_{i=1}^{n+1}$ by extending the segments of the convex and piecewise linear function h_t; see Figure 5.2(b). Firstly, let

$$\left(\check{s}_t^1, \check{y}_t^1\right) := \left(s_t^1, h_t^1\left(s_t^1\right)\right) = \left(s_t^1, h_t^2\left(s_t^1\right)\right), \tag{5.85}$$

which is an intersection of the straight lines correspond to h_t^1 and h_t^2. Similarly, let

$$\left(\check{s}_t^{n+1}, \check{y}_t^{n+1}\right) := \left(s_t^n, h_t^n\left(s_t^n\right)\right) = \left(s_t^n, h_t^{n+1}\left(s_t^n\right)\right), \tag{5.86}$$

which is an intersection of the straight lines correspond to h_t^n and h_t^{n+1}. Notice that

$$[\check{s}_t^1, \check{s}_t^{n+1}] = [s_t^1, s_t^n] = [S_t^b, S_t^a]. \tag{5.87}$$

Secondly, for any $i = 2, \dots, n$, we define $(\check{s}_t^i, \check{y}_t^i)$ as

$$\check{s}_t^i := \begin{cases} -\frac{b_t^{i+1} - b_t^{i-1}}{m_t^{i+1} - m_t^{i-1}} & \text{if } m_t^{i-1} < m_t^{i+1}, \\ \frac{1}{2}\left(s_t^{i-1} + s_t^i\right) & \text{if } m_t^{i-1} = m_t^{i+1}, \end{cases} \tag{5.88}$$

$$\check{y}_t^i := h_t^{i-1}\left(\check{s}_t^i\right). \tag{5.89}$$

Observe that if $m_t^{i-1} = m_t^{i+1}$, then $m_t^{i-1} = m_t^i = m_t^{i+1}$ by (5.84). Then it follows from Remark 5.28 that $h_t^{i-1} = h_t^i = h_t^{i+1}$, and the value $\check{y}_t^i$ defined in (5.89) can be written as

$$\check{y}_t^i = h_t^{i-1}\left(\check{s}_t^i\right) = h_t^i\left(\check{s}_t^i\right) = h_t^{i+1}\left(\check{s}_t^i\right). \tag{5.90}$$

Lemma 5.29 below shows that $(\check{s}_t^i, \check{y}_t^i)$ is an intersection of the straight lines correspond to h_t^{i-1} and h_t^{i+1}, and that it is a point in

$$[s_t^{i-1}, s_t^i] \times \left(-\infty, \check{J}_t^*(\check{s}_t^i)\right].$$

Then the sequence $(\check{s}_t^i)_{i=1}^{n+1}$ is nondecreasing, in other words,

$$\check{s}_t^1 \le \check{s}_t^2 \le \cdots \le \check{s}_t^{n+1}$$

because $\check{s}_t^1 = s_t^1 < s_t^2 < \cdots < s_t^n = \check{s}_t^{n+1}$ and $\check{s}_t^i \in [s_t^{i-1}, s_t^i]$ for all $i = 2, \dots, n$.

Lemma 5.29. *Suppose that $\check{J}_t^*$ is real-valued, continuous, and convex on $[s_t^0, s_t^{n+1}]$. Then for any $i = 2, \dots, n$, the pair $(\check{s}_t^i, \check{y}_t^i)$ defined in* (5.88)-(5.89) *solves*

$$h_t^{i-1}\left(\check{s}_t^i\right) = m_t^{i-1}\check{s}_t^i + b_t^{i-1} = \check{y}_t^i, \tag{5.91}$$

$$h_t^{i+1}\left(\check{s}_t^i\right) = m_t^{i+1}\check{s}_t^i + b_t^{i+1} = \check{y}_t^i, \tag{5.92}$$

and it satisfies $\check{s}_t^i \in [s_t^{i-1}, s_t^i]$ and $\check{y}_t^i \le \check{J}_t^*(\check{s}_t^i)$.

The proof of Lemma 5.29 above will be provided at the end of this section.

Remark 5.30. By letting $i = 2$ in (5.91) and $i = n$ in (5.92), it follows that

$h_t^1(\check{s}_t^2) = \check{y}_t^2$ and $h_t^{n+1}(\check{s}_t^n) = \check{y}_t^n$. Moreover, we have from (5.85) and (5.86) that $h_t^1(\check{s}_t^1) = \check{y}_t^1$ and $h_t^{n+1}(\check{s}_t^{n+1}) = \check{y}_t^{n+1}$. Then

$$\check{s}_t^1 = \check{s}_t^2 \implies \left(\check{s}_t^1, \check{y}_t^1\right) = \left(\check{s}_t^2, \check{y}_t^2\right),$$
$$\check{s}_t^n = \check{s}_t^{n+1} \implies \left(\check{s}_t^n, \check{y}_t^n\right) = \left(\check{s}_t^{n+1}, \check{y}_t^{n+1}\right).$$

In addition, the function h_t^1 corresponds to the straight line crossing $(\check{s}_t^1, \check{y}_t^1)$ and $(\check{s}_t^2, \check{y}_t^2)$. Similarly, the function h_t^{n+1} corresponds to the straight line crossing $(\check{s}_t^n, \check{y}_t^n)$ and $(\check{s}_t^{n+1}, \check{y}_t^{n+1})$.

Remark 5.31. Suppose that $\check{s}_t^k = \check{s}_t^{k+1}$ for some $k = 2, \ldots, n-1$. Then this common value must be s_t^k because $\check{s}_t^k \leq s_t^k \leq \check{s}_t^{k+1}$ (Lemma 5.29). Moreover, by taking $i = k$ in (5.92) and $i = k+1$ in (5.91), it yields $h_t^{k+1}(s_t^k) = \check{y}_t^k$ and $h_t^k(s_t^k) = \check{y}_t^{k+1}$ respectively. It follows from (5.80) that

$$\check{y}_t^k = h_t^{k+1}\left(s_t^k\right) = h_t^k\left(s_t^k\right) = \check{y}_t^{k+1}.$$

Observe that $(\check{s}_t^k, \check{y}_t^k) = (\check{s}_t^{k+1}, \check{y}_t^{k+1})$, and moreover

$$h_t^k\left(\check{s}_t^k\right) = \check{y}_t^k, \qquad h_t^k\left(\check{s}_t^{k+1}\right) = \check{y}_t^{k+1} \tag{5.93}$$

because $s_t^k = \check{s}_t^k = \check{s}_t^{k+1}$.

The last step is to set $\check{J}_t$ to be a continuous and piecewise linear function on the interval $[\check{s}_t^1, \check{s}_t^{n+1}]$ by connecting the points $(\check{s}_t^i, \check{y}_t^i)$ and $(\check{s}_t^{i+1}, \check{y}_t^{i+1})$ for each $i = 1, \ldots, n$; see Figure 5.2(c).

Firstly, for every $i = 1, \ldots, n$, we are going to define $\check{m}_t^i, \check{b}_t^i \in \mathbb{R}$ such that the affine function $\check{h}_t^i : \mathbb{R} \to \mathbb{R}$ of the form

$$\check{h}_t^i(s) = \check{m}_t^i s + \check{b}_t^i$$

satisfies

$$\check{h}_t^i\left(\check{s}_t^i\right) = \check{y}_t^i, \qquad \check{h}_t^i\left(\check{s}_t^{i+1}\right) = \check{y}_t^{i+1}. \tag{5.94}$$

As long as (5.94) holds true, when $(\check{s}_t^i, \check{y}_t^i) \neq (\check{s}_t^{i+1}, \check{y}_t^{i+1})$, the function $\check{h}_t^i$ corresponds to the unique straight line connecting $(\check{s}_t^i, \check{y}_t^i)$ and $(\check{s}_t^{i+1}, \check{y}_t^{i+1})$. Moreover, if (5.94) holds true for all $i = 1, \ldots, n$, then

$$\check{h}_t^1\left(\check{s}_t^2\right) = \check{h}_t^2\left(\check{s}_t^2\right), \check{h}_t^2\left(\check{s}_t^3\right) = \check{h}_t^3\left(\check{s}_t^3\right), \ldots, \check{h}_t^{n-1}\left(\check{s}_t^n\right) = \check{h}_t^n\left(\check{s}_t^n\right).$$

By letting

$$\check{m}_t^1 := m_t^1, \qquad \check{b}_t^1 := b_t^1, \tag{5.95}$$

and

$$\check{m}_t^n := m_t^{n+1}, \qquad \check{b}_t^n := b_t^{n+1}, \tag{5.96}$$

we have $\check{h}_t^1 = h_t^1$ and $\check{h}_t^n = h_t^{n+1}$. Then Remark 5.30 implies that (5.94) is satisfied for each $i = 1, n$. For any $i = 2, \dots, n-1$, we will define $\check{m}_t^i$ and $\check{b}_t^i$ by considering the following two cases. In the case when $\check{s}_t^i < \check{s}_t^{i+1}$, we define

$$\check{m}_t^i := \frac{\check{y}_t^{i+1} - \check{y}_t^i}{\check{s}_t^{i+1} - \check{s}_t^i}, \qquad \check{b}_t^i := \check{y}_t^i - \check{m}_t^i \check{s}_t^i.$$

Then the condition (5.94) is satisfied by straightforward calculation. In the case when $\check{s}_t^i = \check{s}_t^{i+1}$, we take

$$\check{m}_t^i := m_t^i, \qquad \check{b}_t^i := b_t^i. \tag{5.97}$$

This implies $\check{h}_t^i = h_t^i$, and (5.94) follows from (5.93). This completes the constructions of $\check{h}_t^1, \dots, \check{h}_t^n$. Notice that the condition (5.94) holds true for every $i = 1, \dots, n$.

Now, we define $\check{J}_t : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ as

$$\begin{aligned}
&\check{J}_t(s) := \check{h}_t^i(s) \text{ for any } s \in \left[\check{s}_t^i, \check{s}_t^{i+1}\right) \text{ for each } i = 1, \dots, n-1, \\
&\check{J}_t(s) := \check{h}_t^n(s) \text{ for any } s \in \left[\check{s}_t^n, \check{s}_t^{n+1}\right], \\
&\check{J}_t(s) := \infty \text{ for any } s \in \mathbb{R} \backslash \left[\check{s}_t^1, \check{s}_t^{n+1}\right].
\end{aligned} \tag{5.98}$$

Notice that $\check{J}_t$ is real-valued, continuous and piecewise linear on $[\check{s}_t^1, \check{s}_t^{n+1}]$, and it satisfies

$$\check{J}_t = \check{h}_t^i \text{ on } \left[\check{s}_t^i, \check{s}_t^{i+1}\right] \text{ for all } i = 1, \dots, n. \tag{5.99}$$

The following result summarise a number of properties of $\check{J}_t$, and its proof will be provided at the end of this section.

Lemma 5.32. *Suppose that $\check{J}_t^*$ is real-valued, continuous, and convex on $[s_t^0, s_t^{n+1}]$. Then $\check{J}_t$ is $\mathbb{R} \cup \{\infty\}$-valued and convex on $\mathbb{R}$. Moreover, it is continuous, piecewise linear, and satisfies $\check{J}_t \le \check{J}_t^*$ on* $\operatorname{dom} \check{J}_t = [s_t^1, s_t^n] = [S_t^b, S_t^a]$.

Notice that the construction of $\check{J}_t$ is complete. Thus, we have completed the definition of $(\check{J}_t^*)_{t=0}^T$ and $(\check{J}_t)_{t=0}^T$. Proposition 5.33 below provides a number of properties of $(\check{J}_t^*)_{t=0}^T$ and $(\check{J}_t)_{t=0}^T$; Theorem 4.13 established in Sections 4.2 will be used to prove Proposition 5.33.3. In particular, this proposition shows that $\check{J}_t$ is dominated by J_t for all $t = 0, \dots, T$, and this property will be used to compute the approximation error of $(J_t)_{t=0}^T$.

Proposition 5.33. *Let $t = 0, \dots, T$. The following claims hold true.*

1. *For any $\nu \in \Omega_t$, if $t = T$, then $\check{J}^{*\nu}_T$ is affine on* $\operatorname{dom} \check{J}^{*\nu}_T = [S^{b\nu}_T, S^{a\nu}_T]$. *Moreover, if $t \leq T-1$, then $\check{J}^{*\nu}_t$ is $\mathbb{R} \cup \{\infty\}$-valued and convex on $\mathbb{R}$, and it is continuous on* $\operatorname{dom} \check{J}^{*\nu}_t = \left[s^{0\nu}_t, s^{n+1,\nu}_t\right]$.

2. *For any $\nu \in \Omega_t$, the function $\check{J}^{\nu}_t$ is $\mathbb{R} \cup \{\infty\}$-valued and convex on $\mathbb{R}$, and it is continuous and piecewise linear on* $\operatorname{dom} \check{J}^{\nu}_t = [s^{1\nu}_t, s^{n\nu}_t] = [S^{b\nu}_t, S^{a\nu}_t]$. *Moreover, we have $\check{J}^{\nu}_t \leq \check{J}^{*\nu}_t \leq J^{\nu}_t$ on* $\operatorname{dom} \check{J}^{\nu}_t$.

3. *For any $\nu \in \Omega_t$, in the situation when $t \leq T-1$, there exists a solution to the minimisation problem in* (5.76) *for all $s \in$* $\operatorname{dom} \check{J}^{*\nu}_t$.

Proof. Firstly, we are going to prove Claims 1-2 by backward induction. For any $\nu \in \Omega_T$, the function $\check{J}^{\nu}_T = \check{J}^{*\nu}_T = J^{\nu}_T$ is affine on the common effective domain $[S^{b\nu}_T, S^{a\nu}_T]$ by (5.75). This means that Claims 1-2 hold true for $t = T$. Let $k = 0, \ldots, T-1$, and suppose that Claims 1-2 hold true for $t = k+1$. Fix any $\nu \in \Omega_k$. Since Claim 2 holds true for $t = k+1$, for any $\lambda \in \nu^+$ the function $\check{J}^{\lambda}_{k+1}$ is $\mathbb{R} \cup \{\infty\}$-valued and convex on $\mathbb{R}$, and it is continuous on $\operatorname{dom} \check{J}^{\lambda}_{k+1} = [S^{b\lambda}_{k+1}, S^{a\lambda}_{k+1}]$. Moreover, we have $\check{J}^{\lambda}_{k+1} \leq J^{\lambda}_{k+1}$ on $\operatorname{dom} \check{J}^{\lambda}_{k+1}$. Observe from Proposition 5.18.1 that

$$\operatorname{dom} J^{\lambda}_{k+1} = \left[S^{b\lambda}_{k+1}, S^{a\lambda}_{k+1}\right] = \operatorname{dom} \check{J}^{\lambda}_{k+1},$$

and this means that $\check{J}^{\lambda}_{k+1} \leq J^{\lambda}_{k+1}$ always holds true. Then Lemma 5.27 implies that Claim 1 holds true for $t = k$, and

$$\check{J}^{*\nu}_k \leq J^{\nu}_k \text{ on } \left[S^{b\nu}_k, S^{a\nu}_k\right]. \tag{5.100}$$

Notice that $\check{J}^{*\nu}_k$ is real-valued, continuous, and convex on $\left[s^{0\nu}_k, s^{n+1,\nu}_k\right]$. This enable us to prove Claim 2 with $t = k$ by considering the following two cases. In the case when $s^{1\nu}_t = s^{n\nu}_t$, combining (5.77) with (5.100), Claim 2 holds true for $t = k$. In the case when $s^{1\nu}_t < s^{n\nu}_t$, it follows from Lemma 5.32 and (5.100) that Claim 2 holds true for $t = k$. This completes the induction step, and hence Claims 1-2 hold true for all $t = 0, \ldots, T$. Finally, Claim 3 follows from Theorem 4.13. □

This section ends with the proofs of Lemmas 5.27, 5.29, and 5.32.

Proof of Lemma 5.27. Since $\operatorname{dom} \check{J}^{\lambda}_{t+1} = [S^{b\lambda}_{t+1}, S^{a\lambda}_{t+1}]$ for all $\lambda \in \nu^+$, we have

$$\begin{aligned}
\operatorname{co}\left(\bigcup_{\lambda \in \nu^+} \operatorname{dom} \check{J}^{\lambda}_{t+1}\right) &= \operatorname{co}\left(\bigcup_{\lambda \in \nu^+} \left[S^{b\lambda}_{t+1}, S^{a\lambda}_{t+1}\right]\right) \\
&= \left[\min_{\lambda \in \nu^+} S^{b\lambda}_{t+1}, \max_{\lambda \in \nu^+} S^{a\lambda}_{t+1}\right] = \left[s^{0\nu}_t, s^{n+1,\nu}_t\right]
\end{aligned}$$

by (5.73), where $\mathrm{co}\,(A)$ is the convex hull of any set A. From Theorem 4.3, it follows that $\check{J}_t^{*\nu}$ defined in (5.76) is an $\mathbb{R} \cup \{\infty\}$-valued convex function with

$$\mathrm{dom}\, \check{J}_t^{*\nu} = \left[s_t^{0\nu}, s_t^{n+1,\nu}\right].$$

Moreover, the function $\check{J}_t^{*\nu}$ is continuous on $\mathrm{dom}\, \check{J}_t^{*\nu}$ by Theorem 4.13. In addition, if

$$\check{J}_{t+1}^{\lambda} \leq J_{t+1}^{\lambda} \text{ for all } \lambda \in \nu^{+},$$

then it follows from (5.76) and (5.42) that $\check{J}_t^{*\nu} \leq J_t^{\nu}$ on $[S_t^{b\nu}, S_t^{a\nu}]$. This completes the proof. □

Proof of Lemma 5.29. Fix any $i = 2, \ldots, n$. Firstly, we are going to show that $(\check{s}_t^i, \check{y}_t^i)$ defined in (5.88)-(5.89) solves (5.91)-(5.92) by considering the following two cases. In the case when $m_t^{i-1} < m_t^{i+1}$, by straightforward calculation, the pair $(\check{s}_t^i, \check{y}_t^i)$ solves (5.91)-(5.92). In the case when $m_t^{i-1} = m_t^{i+1}$, it follows from (5.90) that

$$\begin{aligned} h_t^{i-1}\left(\check{s}_t^i\right) &= \check{y}_t^i, \\ h_t^{i+1}\left(\check{s}_t^i\right) &= \check{y}_t^i. \end{aligned}$$

Thus $(\check{s}_t^i, \check{y}_t^i)$ always solves (5.91)-(5.92).

Secondly, we are going to prove that

$$\check{s}_t^i \in \left[s_t^{i-1}, s_t^i\right]. \tag{5.101}$$

Notice from (5.84) that

$$m_t^{i-1} \leq m_t^i \leq m^{i+1}.$$

Consider the following two situations for m_t^{i-1} and m_t^{i+1}. In the situation when $m_t^{i-1} = m_t^{i+1}$, we have from (5.88) that

$$\check{s}_t^i = \frac{1}{2}\left(s_t^{i-1} + s_t^i\right) \in \left(s_t^{i-1}, s_t^i\right),$$

and hence (5.101) holds true. In the second situation, we assume $m_t^{i-1} < m_t^{i+1}$, in other words,

$$m_t^{i+1} - m_t^{i-1} > 0.$$

Observe from (5.80) that

$$\begin{aligned} h_t^i\left(s_t^i\right) &= h_t^{i+1}\left(s_t^i\right), \\ h_t^i\left(s_t^{i-1}\right) &= h_t^{i-1}\left(s_t^{i-1}\right). \end{aligned}$$

Then the slope m_t^i of the affine function h_t^i can be written as

$$\begin{aligned} m_t^i &= \frac{h_t^i\left(s_t^i\right) - h_t^i\left(s_t^{i-1}\right)}{s_t^i - s_t^{i-1}} \\ &= \frac{h_t^{i+1}\left(s_t^i\right) - h_t^{i-1}\left(s_t^{i-1}\right)}{s_t^i - s_t^{i-1}} \\ &= \frac{m_t^{i+1}s_t^i + b_t^{i+1} - m_t^{i-1}s_t^{i-1} - b_t^{i-1}}{s_t^i - s_t^{i-1}}. \end{aligned} \tag{5.102}$$

From $m_t^{i-1} \leq m_t^i$ and (5.102), we have

$$m_t^{i-1} \leq \frac{m_t^{i+1}s_t^i + b_t^{i+1} - m_t^{i-1}s_t^{i-1} - b_t^{i-1}}{s_t^i - s_t^{i-1}},$$

which implies

$$s_t^i \geq -\frac{b_t^{i+1} - b_t^{i-1}}{m_t^{i+1} - m_t^{i-1}} = \check{s}_t^i$$

by (5.88). Similarly, combining $m_t^{i+1} \geq m_t^i$ and (5.102), it follows that

$$m_t^{i+1} \geq \frac{m_t^{i+1}s_t^i + b_t^{i+1} - m_t^{i-1}s_t^{i-1} - b_t^{i-1}}{s_t^i - s_t^{i-1}}$$

which means

$$s_t^{i-1} \leq -\frac{b_t^{i+1} - b_t^{i-1}}{m_t^{i+1} - m_t^{i-1}} = \check{s}_t^i$$

by (5.88) again. Thus (5.101) always holds true.

Finally, we are going to show that $\check{y}_t^i \leq \check{J}_t^*(\check{s}_t^i)$. Observe from (5.78)-(5.79) that

$$\begin{aligned} h_t^{i-1}\left(s_t^{i-2}\right) &= \check{J}_t^*\left(s_t^{i-2}\right), \\ h_t^{i-1}\left(s_t^{i-1}\right) &= \check{J}_t^*\left(s_t^{i-1}\right). \end{aligned}$$

Moreover, we have $s_t^{i-1} \leq \check{s}_t^i$ by (5.101). Then it follows from (5.91) and Lemma A.8 that

$$\check{y}_t^i = h_t^{i-1}(\check{s}_t^i) \leq \check{J}_t^*(\check{s}_t^i).$$

This completes the proof. □

Proof of Lemma 5.32. Observe from the comments preceding (5.99) that $\check{J}_t$ is real-valued, continuous and piecewise linear on $[\check{s}_t^1, \check{s}_t^{n+1}]$; the interval $[\check{s}_t^1, \check{s}_t^{n+1}]$ can be written as $[\check{s}_t^1, \check{s}_t^{n+1}] = [s_t^1, s_t^n] = [S_t^b, S_t^a]$ by (5.87). Moreover, it follows from (5.98) that $\check{J}_t = \infty$ on $\mathbb{R}\backslash[\check{s}_t^1, \check{s}_t^{n+1}]$, and therefore it is enough to show that $\check{J}_t \le \check{J}_t^*$ on $[\check{s}_t^1, \check{s}_t^{n+1}]$ and $\check{J}_t$ is convex.

The relationship between $\check{J}_t$ and $\check{h}_t^1, \ldots, \check{h}_t^n$ in (5.99) implies that $\check{J}_t \le \check{J}_t^*$ on $[\check{s}_t^1, \check{s}_t^{n+1}]$ if and only if

$$\check{h}_t^i \le \check{J}_t^* \text{ on } \left[\check{s}_t^i, \check{s}_t^{i+1}\right] \tag{5.103}$$

for all $i = 1, \ldots, n$. Observe from (5.78)-(5.79) that, for any $k = 1, \ldots, n+1$, the affine function h_t^k satisfies

$$h_t^k(s) = \check{J}_t^*(s) \text{ for each } s = s_t^{k-1}, s_t^k,$$

where h_t^k is affine and $\check{J}_t^*$ is convex. Then Lemma A.8 gives

$$h_t^k \le \check{J}_t^* \text{ on } \left[s_t^0, s_t^{n+1}\right] \setminus \left(s_t^{k-1}, s_t^k\right). \tag{5.104}$$

Firstly, we are going to prove that (5.103) holds true for each $k = 1, n$ respectively. By taking $k = 1$ in (5.104), it follows from $s_t^k = s_t^1 = \check{s}_t^1 \le \check{s}_t^2$ that $h_t^1 \le \check{J}_t^*$ on $[\check{s}_t^1, \check{s}_t^2]$. Similarly, by taking $k = n+1$ in (5.104), we have $h_t^{n+1} \le \check{J}_t^*$ on $[\check{s}_t^n, \check{s}_t^{n+1}]$ because $\check{s}_t^n \le \check{s}_t^{n+1} = s_t^n = s_t^{k-1}$. Observe from the comments following (5.96) that $\check{h}_t^1 = h_t^1$ and $\check{h}_t^n = h_t^{n+1}$. Thus (5.103) holds true for each $i = 1, n$.

To complete the proof of $\check{J}_t \le \check{J}_t^*$ on $[\check{s}_t^1, \check{s}_t^{n+1}]$, we are going to show that (5.103) holds true for every $i = 2, \ldots, n-1$. For each $i = 2, \ldots, n-1$, we define a continuous piecewise linear function $F^i : [\check{s}_t^i, \check{s}_t^{i+1}] \to \mathbb{R}$ with two segments as

$$F^i(s) = \begin{cases} h_t^{i+1}(s) & \text{if } s \in [\check{s}_t^i, s_t^i], \\ h_t^i(s) & \text{if } s \in \left[s_t^i, \check{s}_t^{i+1}\right], \end{cases}$$

where $\check{s}_t^i \le s_t^i \le \check{s}_t^{i+1}$ by Lemma 5.29, and $h_t^{i+1}(s_t^i) = h_t^i(s_t^i)$ by (5.80). Notice that $-F^i$ is convex because the slopes of the affine functions $-h_t^{i+1}$ and $-h_t^i$ satisfies $-m_t^{i+1} \le -m_t^i$ (see (5.84)). Moreover, it follows from (5.104) that $h_t^{i+1} \le \check{J}_t^*$ on $[\check{s}_t^i, s_t^i]$ and $h_t^i \le \check{J}_t^*$ on $[s_t^i, \check{s}_t^{i+1}]$, and hence

$$F^i \le \check{J}_t^* \text{ on } \left[\check{s}_t^i, \check{s}_t^{i+1}\right].$$

Observe that $h_t^{i+1}(\check{s}_t^i) = \check{y}_t^i$ by (5.92), and $h_t^i(\check{s}_t^{i+1}) = \check{y}_t^{i+1}$ by (5.91). Combining this with (5.94), it follows that

$$h_t^{i+1}\left(\check{s}_t^i\right) = \check{h}_t^i\left(\check{s}_t^i\right), \tag{5.105}$$

$$h_t^i\left(\check{s}_t^{i+1}\right) = \check{h}_t^i\left(\check{s}_t^{i+1}\right). \tag{5.106}$$

This leads to $F^i(\check{s}_t^i) = \check{h}_t^i(\check{s}_t^i)$ and $F^i(\check{s}_t^{i+1}) = \check{h}_t^i(\check{s}_t^{i+1})$. Consider the following two cases of the interval $[\check{s}_t^i, \check{s}_t^{i+1}]$. In the case when $\check{s}_t^i = \check{s}_t^{i+1}$ (i.e. $[\check{s}_t^i, \check{s}_t^{i+1}]$ is a singleton set), we have $F^i = \check{h}_t^i$ on $[\check{s}_t^i, \check{s}_t^{i+1}]$. In the case when $\check{s}_t^i < \check{s}_t^{i+1}$, from Lemma A.8, we have $-\check{h}_t^i \geq -F^i$ on $[\check{s}_t^i, \check{s}_t^{i+1}]$. Thus

$$\check{h}_t^i \leq F^i \leq \check{J}_t^* \text{ on } \left[\check{s}_t^i, \check{s}_t^{i+1}\right].$$

This completes the proof of (5.103) for all $i = 1, \dots, n$. Therefore, we can conclude that $\check{J}_t \leq \check{J}_t^*$ on $[\check{s}_t^1, \check{s}_t^{n+1}]$.

Finally, we are going to prove that $\check{J}_t$ is convex. By the connection between $\check{J}_t$ and $\check{h}_t^1, \dots, \check{h}_t^n$ in (5.99), the function $\check{J}_t$ is convex if the slopes of $\check{h}_t^1, \dots, \check{h}_t^n$ satisfy

$$\check{m}_t^1 \leq \cdots \leq \check{m}_t^n \tag{5.107}$$

(Lemma A.9.2). To prove (5.107), by (5.84), it is enough to show that

$$\check{m}_t^i \in \left[m_t^i, m_t^{i+1}\right] \tag{5.108}$$

for all $i = 1, \dots, n$. Notice that $\check{m}_t^1 = m_t^1$ and $\check{m}_t^n = m_t^{n+1}$ by (5.95)-(5.96). Thus (5.108) holds true for each $i = 1, n$. Fix any $i = 2, \dots, n-1$. By Lemma 5.29, we have $\check{s}_t^i \leq s_t^i \leq \check{s}_t^{i+1}$. We are going to prove (5.108) by considering the following two situations for the interval $[\check{s}_t^i, \check{s}_t^{i+1}]$.

1. In the situation when $\check{s}_t^i = \check{s}_t^{i+1}$, we have from (5.97) that $\check{m}_t^i = m_t^i$, and hence (5.108) holds true.

2. In the situation when $\check{s}_t^i < \check{s}_t^{i+1}$, we are going to prove (5.108) by showing $m_t^i \leq \check{m}_t^i$ and $m_t^{i+1} \geq \check{m}_t^i$ respectively. The slopes of the affine functions $h_t^1, \dots, h_t^{n+1}$ satisfy (5.84), and these functions relate to h_t by (5.83), where h_t defined in (5.81)-(5.82) is continuous and piecewise linear on $[s_t^0, s_t^{n+1}]$. We have from (5.83) that $h_t = h_t^i$ on $[s_t^{i-1}, s_t^i]$ which contains $\check{s}_t^i$ (Lemma 5.29). Then Lemma A.9.1 gives

$$h_t^i\left(\check{s}_t^i\right) \geq h_t^{i+1}\left(\check{s}_t^i\right).$$

Similarly, we have $h_t = h_t^{i+1}$ on $[s_t^i, s_t^{i+1}]$ which contains $\check{s}_t^{i+1}$ (Lemma 5.29). By Lemma A.9.1 again, it follows that

$$h_t^{i+1}\left(\check{s}_t^{i+1}\right) \geq h_t^i\left(\check{s}_t^{i+1}\right).$$

Then (5.105) and (5.106) implies respectively that

$$h_t^i\left(\check{s}_t^i\right) \geq \check{h}_t^i\left(\check{s}_t^i\right), \tag{5.109}$$

$$h_t^{i+1}\left(\check{s}_t^{i+1}\right) \geq \check{h}_t^i\left(\check{s}_t^{i+1}\right). \tag{5.110}$$

Observe from (5.106) and (5.109) that

$$m_t^i = \frac{h_t^i\left(\check{s}_t^{i+1}\right) - h_t^i\left(\check{s}_t^i\right)}{\check{s}_t^{i+1} - \check{s}_t^i} \leq \frac{\check{h}_t^i\left(\check{s}_t^{i+1}\right) - \check{h}_t^i\left(\check{s}_t^i\right)}{\check{s}_t^{i+1} - \check{s}_t^i} = \check{m}_t^i.$$

Moreover, it follows from (5.110) and (5.105) that

$$m_t^{i+1} = \frac{h_t^{i+1}\left(\check{s}_t^{i+1}\right) - h_t^{i+1}\left(\check{s}_t^i\right)}{\check{s}_t^{i+1} - \check{s}_t^i} \geq \frac{\check{h}_t^i\left(\check{s}_t^{i+1}\right) - \check{h}_t^i\left(\check{s}_t^i\right)}{\check{s}_t^{i+1} - \check{s}_t^i} = \check{m}_t^i.$$

Thus (5.108) holds true.

Therefore (5.107) holds true and $\check{J}_t$ is convex. This completes the proof. □

5.5 Numerical examples in a binomial model

Consider the friction-free Cox-Ross-Rubinstein (CRR) binomial model with parameters u, r and d such that

$$1 + \mathrm{u} = e^{\sigma\sqrt{\frac{1}{T}}}, \tag{5.111}$$

$$1 + \mathrm{r} = (1 + r_e)^{\frac{1}{T}}, \tag{5.112}$$

$$1 + \mathrm{d} = e^{-\sigma\sqrt{\frac{1}{T}}}, \tag{5.113}$$

where $\sigma = 0.2$ is used to model the volatility of the return on stock per annum, and $r_e > -1$ is the annually compounded interest rate. We are using this discrete-time model with number of steps T to approximate a continuous-time model with horizon 1. For every $t = 0, \ldots, T-1$ and $\nu \in \Omega_t$, the node ν has two successor nodes νu and νd, in other words,

$$\nu^+ = \{\nu u, \nu d\}. \tag{5.114}$$

The transition probabilities from ν to νu and from ν to νd are given respectively by

$$p_{t+1}^{\nu u} = \frac{\mathbb{P}(\nu u)}{\mathbb{P}(\nu)} = p, \tag{5.115}$$

$$p_{t+1}^{\nu d} = \frac{\mathbb{P}(\nu d)}{\mathbb{P}(\nu)} = 1 - p, \tag{5.116}$$

where $p \in (0,1)$. Moreover, the friction-free stock price satisfies

$$S_{t+1}^{\nu u} = (1+\mathrm{u})S_t^{\nu}$$
$$S_{t+1}^{\nu d} = (1+\mathrm{d})S_t^{\nu}$$

with $S_0 = 100$ given. The pricing results established in this thesis are based on discounted asset prices, and we are going to construct $(S_t^b, S_t^a)_{t=0}^T$ based on the discounted asset prices in this CRR model. We denote the discounted stock price by

$$\bar{S}_t = \frac{S_t}{(1+\mathrm{r})^t} \text{ for all } t = 0, \dots, T.$$

Notice that $\bar{S}_0 = S_0$ and $\bar{S}_T = \frac{S_T}{1+r_e}$ because $(1+\mathrm{r})^T = 1 + r_e$. Given a transaction cost parameter $k \in [0,1)$, at time $t = 1, \dots, T$, we define the bid and ask prices of the stock as

$$S_t^a = (1+k)\bar{S}_t,$$
$$S_t^b = (1-k)\bar{S}_t.$$

Moreover, for convenience, we assume $S_0^a = S_0^b = \bar{S}_0$, in other words, there are no transaction costs at time 0. Observe that the market model $(S_t^b, S_t^a)_{t=0}^T$ depends on r_e and k. In most examples, the parameter r_e will be zero, except in Example 5.48, we provide the indifference prices for different values of r_e. The theorem below shows that, as long as $\mathrm{d} < \mathrm{r} < \mathrm{u}$, the condition (5.26) holds true, which means that the robust no-arbitrage condition also holds true; see Theorem 5.13. In all numerical examples, the condition $\mathrm{d} < \mathrm{r} < \mathrm{u}$ will be satisfied.

Theorem 5.34. *If* $\mathrm{d} < \mathrm{r} < \mathrm{u}$, *the market model* $(S_t^b, S_t^a)_{t=0}^T$ *satisfies* (5.26).

Proof. Suppose that $\mathrm{d} < \mathrm{r} < \mathrm{u}$. For any $t = 0, \dots, T-1$ and $\nu \in \Omega_t$, we have

$$S_{t+1}^{b\nu d} = (1-k)\bar{S}_{t+1}^{\nu d} = \frac{(1-k)S_{t+1}^{\nu d}}{(1+\mathrm{r})^{t+1}} = \frac{(1+\mathrm{d})(1-k)S_t^{\nu}}{(1+\mathrm{r})^{t+1}} = \frac{1+\mathrm{d}}{1+\mathrm{r}}S_t^{b\nu} < S_t^{b\nu}.$$
$$S_{t+1}^{a\nu u} = (1+k)\bar{S}_{t+1}^{\nu u} = \frac{(1+k)S_{t+1}^{\nu u}}{(1+\mathrm{r})^{t+1}} = \frac{(1+\mathrm{u})(1+k)S_t^{\nu}}{(1+\mathrm{r})^{t+1}} = \frac{1+\mathrm{u}}{1+\mathrm{r}}S_t^{a\nu} > S_t^{a\nu},$$

Combining this with $S_{t+1}^{b\nu d} = \min_{\lambda \in \nu^+} S_{t+1}^{b\lambda}$ and $S_{t+1}^{a\nu u} = \max_{\lambda \in \nu^+} S_{t+1}^{a\lambda}$, the condition (5.26) is satisfied. □

With the exponential regret functions defined in (5.1), the value of $K_{\mathcal{I}}$ defined in (5.7) is important for computing the regret indifference prices defined in (3.51)-(3.52); see Theorem 5.7. Moreover, the value of $K_{\mathcal{I}}$ is used to compute the value of $\hat{\lambda}$ defined in (5.11), and $\hat{\lambda}$ is essential for the calculation of the optimal value and the solution to the problem (3.19); see Theorems 5.5 and 5.6. Let $X \in \mathcal{L}^2_T$, and we are going to introduce a method to approximate $K_{\mathcal{I}}(X)$. It follows from Theorem 5.20 together with $S^a_0 = S^b_0 = S_0$ that $K_{\mathcal{I}}(X)$ can be written as

$$K_{\mathcal{I}}(X) = \inf_{s \in [S^b_0, S^a_0]} J_0(s) = J_0(S_0), \tag{5.117}$$

where $(J_t)_{t=0}^T$ is a sequence of random functions constructed in Algorithm 5.17 from the terminal value

$$J_T(s) = \begin{cases} (1,s)\cdot X & \text{if } S^b_T \leq s \leq S^a_T, \\ \infty & \text{otherwise.} \end{cases}$$

In order to approximate $K_{\mathcal{I}}(X)$, we shall always use the piecewise linear approximation $(\widetilde{J}_t)_{t=0}^T$ introduced in Section 5.3 to approximate $(J_t)_{t=0}^T$. At time T, we have $\widetilde{J}_T = J_T$ by (5.59). At time $t = 0, \ldots, T-1$ and node $\nu \in \Omega_t$, the function $\widetilde{J}^\nu_t$ is a piecewise linear approximation of J^ν_t with $\widetilde{N} := n - 1$ segments, where $n-1$ is the number of intervals in (5.54). Moreover, it follows from (5.58) and Theorem 5.25 that $\widetilde{J}^\nu_t$ converges uniformly to J^ν_t as $\widetilde{N} \to \infty$. By (5.117), we have $K_{\mathcal{I}}(X) = J_0(S_0)$, and this motivates us to approximate $K_{\mathcal{I}}(X)$ by

$$\widetilde{K_{\mathcal{I}}}(X) := \widetilde{J}_0(S_0). \tag{5.118}$$

Since $\widetilde{J}_0(S_0)$ depends on $\widetilde{N}$, the quantity $\widetilde{K_{\mathcal{I}}}(X)$ also depends on $\widetilde{N}$, and

$$\lim_{\widetilde{N}\to\infty} \widetilde{K_{\mathcal{I}}}(X) = \lim_{\widetilde{N}\to\infty} \widetilde{J}_0(S_0) = J_0(S_0) = K_{\mathcal{I}}(X). \tag{5.119}$$

Observe from $\widetilde{J}_0(S_0) \geq J_0(S_0)$ (Proposition 5.24.1) that

$$\widetilde{K_{\mathcal{I}}}(X) \geq K_{\mathcal{I}}(X) \tag{5.120}$$

Thus $\widetilde{K_{\mathcal{I}}}(X)$ converges to $K_{\mathcal{I}}(X)$ from above. We will discuss the performance of this approximation in the next section.

We are now going to introduce the approximation of indifference prices based on $\widetilde{K_{\mathcal{I}}}$. For any $c = (c_t)_{t=0}^T, \bar{c} = (\bar{c}_t)_{t=0}^T \in \mathcal{N}^2$, let

$$\widetilde{\pi}^{ai}_{\mathrm{F}}(c;\bar{c}) := \widetilde{K_{\mathcal{I}}}\left(\textstyle\sum_{t=0}^T \bar{c}_t\right) - \widetilde{K_{\mathcal{I}}}\left(\textstyle\sum_{t=0}^T (\bar{c}_t - c_t)\right) \tag{5.121}$$

and

$$\widetilde{\pi}^{bi}_{\mathrm{F}}(c;\bar{c}) := \widetilde{K_{\mathcal{I}}}\left(\textstyle\sum_{t=0}^T (\bar{c}_t + c_t)\right) - \widetilde{K_{\mathcal{I}}}\left(\textstyle\sum_{t=0}^T \bar{c}_t\right) \tag{5.122}$$

(cf. Theorem 5.7). We will use $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;\bar{c})$ and $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;\bar{c})$ to approximate the indifference prices $\pi^{ai}_{\mathrm{F}}(c;\bar{c})$ and $\pi^{bi}_{\mathrm{F}}(c;\bar{c})$ respectively. By Theorem 5.7 together with (5.119), the quantities $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;\bar{c})$ and $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;\bar{c})$ converge respectively to $\pi^{ai}_{\mathrm{F}}(c;\bar{c})$ and $\pi^{bi}_{\mathrm{F}}(c;\bar{c})$ as $\widetilde{N}\to\infty$. In Section 5.5.3, we will provide numerical examples to compute $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;\bar{c})$ and $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;\bar{c})$. Notice that $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;\bar{c})$ and $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;\bar{c})$ only relies on $\sum_{t=0}^{T} c_t$ and $\sum_{t=0}^{T}\bar{c}_t$. Thus, to calculate $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;\bar{c})$ and $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;\bar{c})$, as long as $\sum_{t=0}^{T} c_t$ and $\sum_{t=0}^{T}\bar{c}_t$ are given, it is not required to know every random variables $c_0,\dots,c_T,\bar{c}_0,\dots,\bar{c}_T$.

5.5.1 Approximation error

In this section, we will compute the values of $(\widetilde{J}_t)_{t=0}^{T}$ and $(\check{J}_t)_{t=0}^{T}$ introduced in Sections 5.3 and 5.4 respectively. Firstly, we will provide examples to plot $\widetilde{J}^{\nu}_t$ and $\check{J}^{\nu}_t$ at some time $t=0,\dots,T-1$ and node $\nu\in\Omega_t$. Secondly, by taking $\check{K}_{\mathcal{I}}(X) := \check{J}_0(S_0)$, we will compute the value $\widetilde{K_{\mathcal{I}}}(X)-\check{K}_{\mathcal{I}}(X)$ in numerical examples, where $\widetilde{K_{\mathcal{I}}}(X)-\check{K}_{\mathcal{I}}(X)$ is an upper bound of the approximation error $\widetilde{K_{\mathcal{I}}}(X)-K_{\mathcal{I}}(X)$. Based on the values of $\widetilde{K_{\mathcal{I}}}-\check{K}_{\mathcal{I}}$, we will present an upper bound for $|\widetilde{\pi}^{ai}_{\mathrm{F}}-\pi^{ai}_{\mathrm{F}}|$ and an upper bound for $|\widetilde{\pi}^{bi}_{\mathrm{F}}-\pi^{bi}_{\mathrm{F}}|$ in Theorem 5.36.

In Sections 5.3 and 5.4, the sequences of random functions $(\widetilde{J}_t)_{t=0}^{T}$ and $(\check{J}_t)_{t=0}^{T}$ are constructed respectively by backward induction with a common terminal value

$$\widetilde{J}_T(s)=\check{J}_T(s)=J_T(s)=\begin{cases}(1,s)\cdot X & \text{if } S^b_T\le s\le S^a_T,\\ \infty & \text{otherwise.}\end{cases}$$

Notice from the comments preceding (5.118) that $(\widetilde{J}_t)_{t=0}^{T}$ depends on the integer $\widetilde{N}$ which is the number of segments used in the piecewise linear approximation. Similarly, the sequence of random functions $(\check{J}_t)_{t=0}^{T}$ depends on the integer $\check{N}:=n-1$ which is the number of intervals in (5.72). Moreover, for any $t=0,\dots,T-1$ and $\nu\in\Omega_t$, the function $\check{J}^{\nu}_t$ is a piecewise linear function with $\check{N}+1$ segments, and it is dominated by J^{ν}_t (Proposition 5.33.2).

Let

$$\mathrm{Call}^{\mathrm{P}}_T(A) := \left(-A\mathbf{1}_{\{S_T>A\}},\mathbf{1}_{\{S_T>A\}}\right)\ \text{for all } A\ge 0, \tag{5.123}$$

where $\mathrm{Call}^{\mathrm{P}}_T(A)$ is the payoff of a call option delivered by portfolio with strike price A, and “P” in the superscript stands for portfolio delivery. In the following example, we will plot $\widetilde{J}^{\nu}_t$ and $\check{J}^{\nu}_t$ at some time $t=0,\dots,T-1$ and node $\nu\in\Omega_t$.

Example 5.35. Let $T=52$, $r_e=0\%$, $p=0.5$, $\mathcal{I}=\{0,\dots,T\}$, and $\alpha_t=1$ for all $t\in\mathcal{I}$. Moreover, we set $\widetilde{N}=120$ and $\check{N}=200$. In addition, let $\nu\in\Omega_{19}$

be the node such that

$$\bar{S}^{\nu}_{19} = S^{\nu}_{19} = S_0 e^{6\sigma\sqrt{\frac{1}{T}}} e^{-13\sigma\sqrt{\frac{1}{T}}} = 82.35384.$$

In Figure 5.3, we plot $\widetilde{J}^{\nu}_{19}$ and $\check{J}^{\nu}_{19}$ on their common effective domain

$$\left[S^{b\nu}_{19}, S^{a\nu}_{19}\right] = \left[(1-k)\bar{S}^{\nu}_{19}, (1+k)\bar{S}^{\nu}_{19}\right]$$

for $k = 0.5\%$ and various X. In Figures 5.3(a)-5.3(c), we set X to be 0, $\mathrm{Call}^{\mathrm{P}}_{T}(100)$, and $-\mathrm{Call}^{\mathrm{P}}_{T}(100)$ respectively. It shows that $\widetilde{J}^{\nu}_{19}$ and $\check{J}^{\nu}_{19}$ are convex, and they are not necessarily monotonic; see Figure 5.3(b). Moreover, the values of $\widetilde{J}^{\nu}_{19}$ and $\check{J}^{\nu}_{19}$ are extremely close. This shows the accuracy of using $\widetilde{J}^{\nu}_{19}$ to approximate J^{ν}_{19} because their values satisfies $\check{J}^{\nu}_{19} \leq J^{\nu}_{19} \leq \widetilde{J}^{\nu}_{19}$. For comparison, we provide $\widetilde{J}^{\nu}_{19}$ and $\check{J}^{\nu}_{19}$ in Figure 5.4 for the increased transaction costs value $k = 3\%$. Figures 5.4(a)-5.4(b) show that there is an obvious gap between $\widetilde{J}^{\nu}_{19}$ and $\check{J}^{\nu}_{19}$. Moreover, comparing to Figure 5.3, the curves in Figure 5.4 tend to be more flat at the bottom.

Figure 5.5 contains plots of $\widetilde{J}^{\nu}_{19}$, $\widetilde{J}^{\nu u}_{20}$, and $\widetilde{J}^{\nu d}_{20}$ on their effective domains for $k = 0.5\%$, where $\{\nu u, \nu d\}$ are the collection of successor nodes of ν (see (5.114)). The function $\widetilde{J}^{\nu}_{19}$ is constructed from $\widetilde{J}^{\nu u}_{20}$ and $\widetilde{J}^{\nu d}_{20}$. Moreover, on $\operatorname{dom} \widetilde{J}^{\nu}_{19} = [S^{\nu b}_{19}, S^{\nu a}_{19}]$, the function $\widetilde{J}^{\nu}_{19}$ can be regarded as a "twisted convex hull" of $\widetilde{J}^{\nu u}_{20}$ and $\widetilde{J}^{\nu d}_{20}$. Observe that there is no overlap between the effective domains when $k = 0.5\%$. The functions $\widetilde{J}^{\nu}_{19}$, $\widetilde{J}^{\nu u}_{20}$, and $\widetilde{J}^{\nu d}_{20}$ for the increased value $k = 3\%$ are provided in Figure 5.6. Due to the larger size of transaction costs, there are overlaps between their effective domains. All curves in Figure 5.6 tend to be more flat at the bottom when compared to their counterparts in Figure 5.5.

Now, we define

$$\check{K}_{\mathcal{I}}(X) := \check{J}_0(S_0) \tag{5.124}$$

which is a lower bound for $K_{\mathcal{I}}(X)$ because $\check{J}_0(S_0) \leq J_0(S_0) = K_{\mathcal{I}}(X)$ by Proposition 5.33.2 and (5.117). Combining (5.120) with $\check{K}_{\mathcal{I}}(X) \leq K_{\mathcal{I}}(X)$, we can conclude that

$$0 \leq \widetilde{K_{\mathcal{I}}}(X) - K_{\mathcal{I}}(X) \leq \widetilde{K_{\mathcal{I}}}(X) - \check{K}_{\mathcal{I}}(X). \tag{5.125}$$

This means that $\widetilde{K_{\mathcal{I}}}(X) - \check{K}_{\mathcal{I}}(X)$ is an upper bound of the approximation error $\widetilde{K_{\mathcal{I}}}(X) - K_{\mathcal{I}}(X)$. The following result gives an error bound of the approximations $\widetilde{\pi}^{ai}_{\mathrm{F}}$ and $\widetilde{\pi}^{bi}_{\mathrm{F}}$ (defined in (5.121) and (5.122)) of π^{ai}_{F} and π^{bi}_{F}.

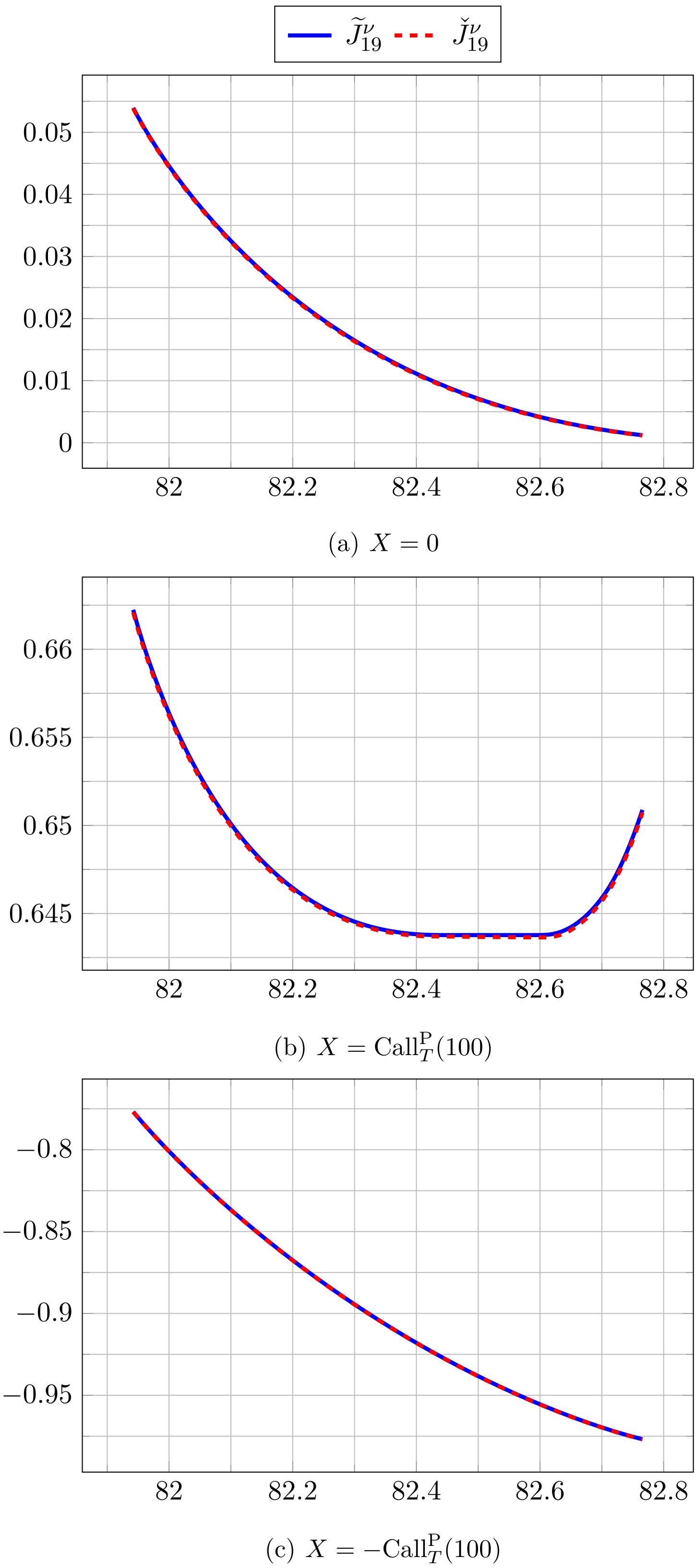
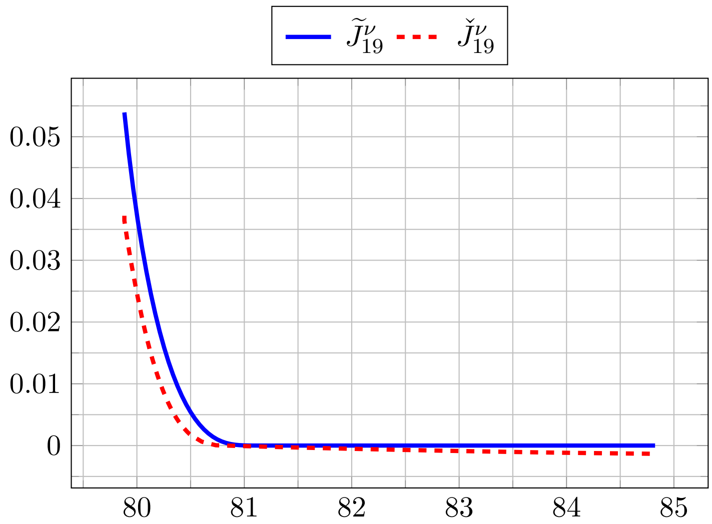
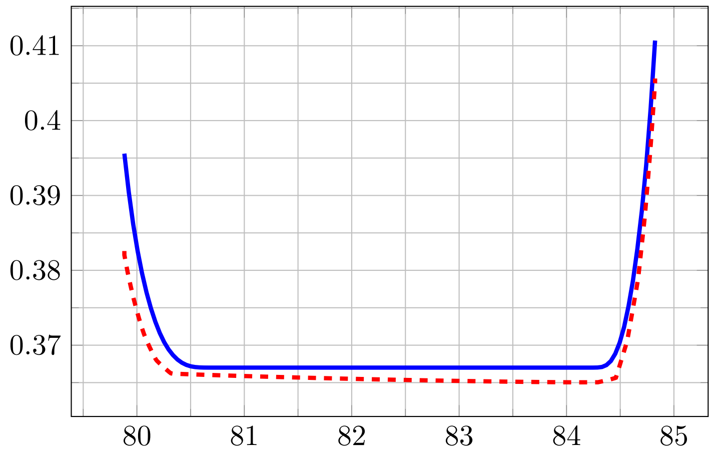


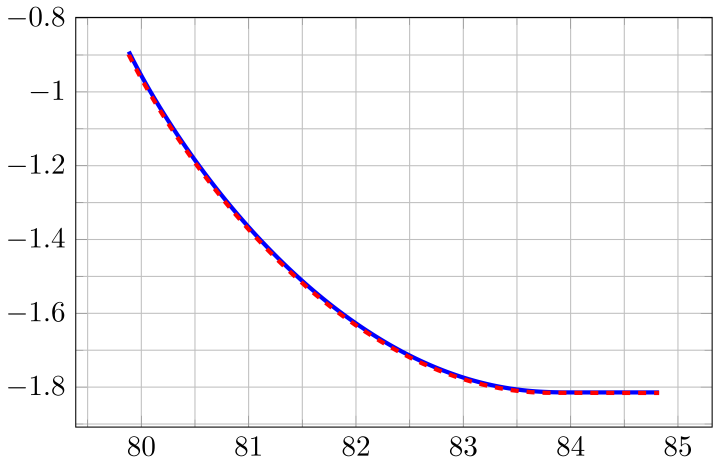
Figure 5.3: The values of $\widetilde{J}^{\nu}_{19}$ and $\check{J}^{\nu}_{19}$ on $\left[S^{b\nu}_{19}, S^{a\nu}_{19}\right]$ with $k = 0.5\%$



(a) $X = 0$

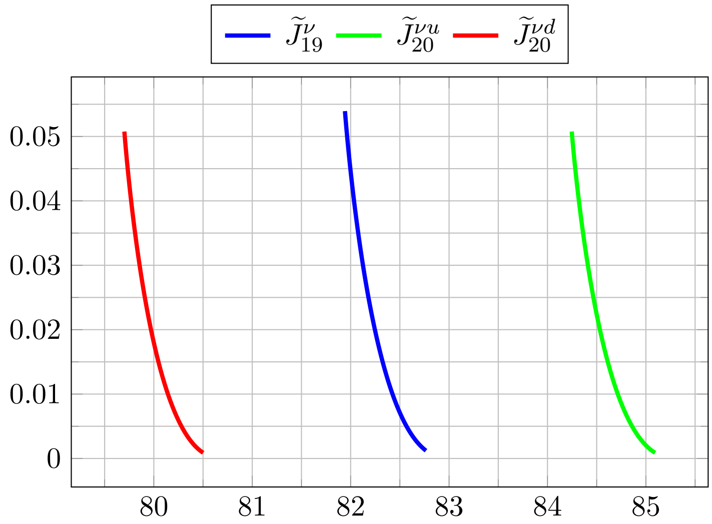


(b) $X = \mathrm{Call}^{\mathrm{P}}_{T}(100)$

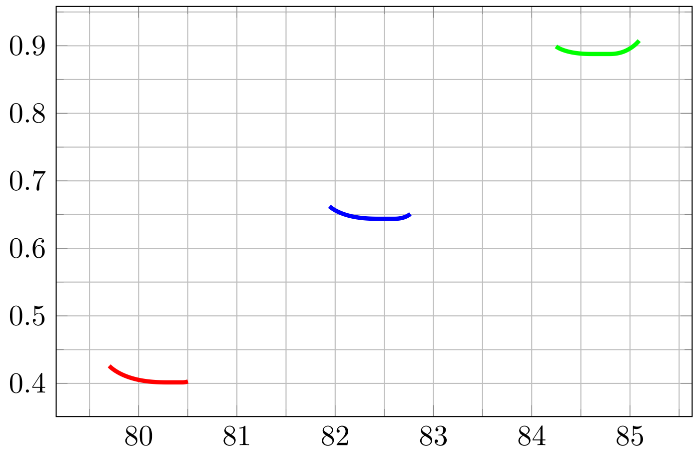


(c) $X = -\mathrm{Call}^{\mathrm{P}}_{T}(100)$

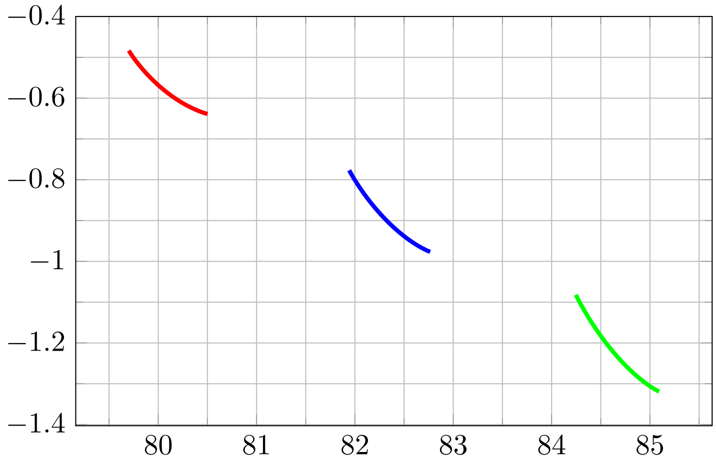
Figure 5.4: The values of $\widetilde{J}^{\nu}_{19}$ and $\check{J}^{\nu}_{19}$ on $\left[S^{b\nu}_{19}, S^{a\nu}_{19}\right]$ with $k = 3\%$



(a) $X = 0$

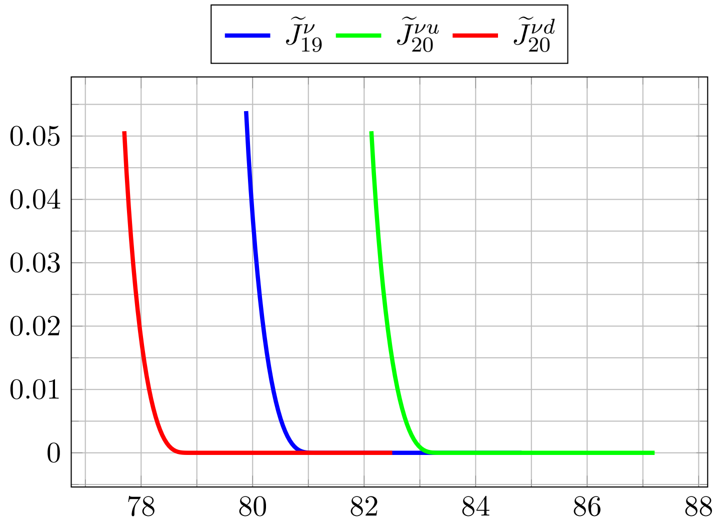


(b) $X = \mathrm{Call}^{\mathrm{P}}_T(100)$

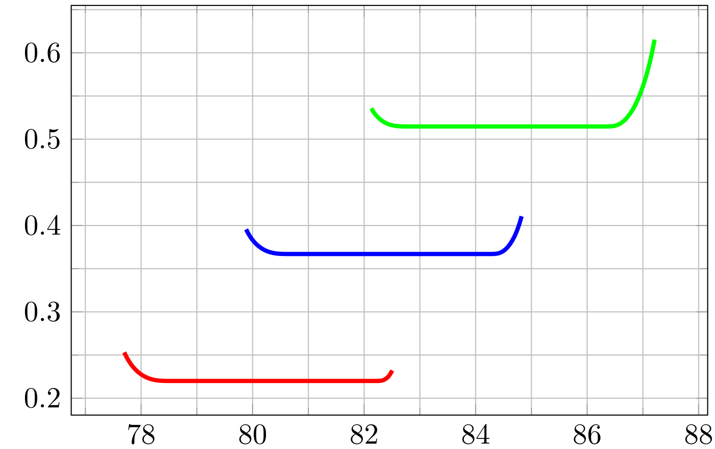


(c) $X = -\mathrm{Call}^{\mathrm{P}}_T(100)$

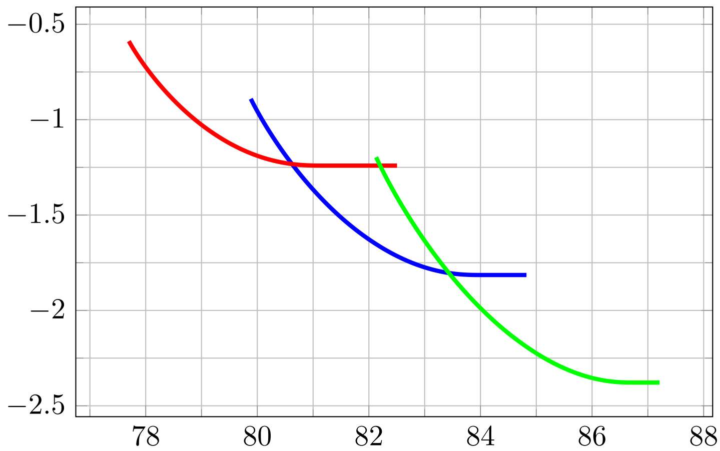
Figure 5.5: The values of $\widetilde{J}^{\nu}_{19}$, $\widetilde{J}^{\nu u}_{20}$ and $\widetilde{J}^{\nu d}_{20}$ on their effective domains with $k = 0.5\%$



(a) $X = 0$



(b) $X = \mathrm{Call}^{\mathrm{P}}_{T}(100)$



(c) $X = -\mathrm{Call}^{\mathrm{P}}_{T}(100)$

Figure 5.6: The values of $\widetilde{J}^{\nu}_{19}$, $\widetilde{J}^{\nu u}_{20}$ and $\widetilde{J}^{\nu d}_{20}$ on their effective domains with $k = 3\%$

Theorem 5.36. *Let* $c = (c_t)_{t=0}^T, \bar{c} = (\bar{c}_t)_{t=0}^T \in \mathcal{N}^2$, *and let*

$$\begin{aligned} \bar{X} &:= \textstyle\sum_{t=0}^T \bar{c}_t, \\ X^{ai} &:= \textstyle\sum_{t=0}^T (\bar{c}_t - c_t), \\ X^{bi} &:= \textstyle\sum_{t=0}^T (\bar{c}_t + c_t). \end{aligned}$$

Then

$$\left|\widetilde{\pi}_{\mathrm{F}}^{ai}(c;\bar{c}) - \pi_{\mathrm{F}}^{ai}(c;\bar{c})\right| \leq \max\left\{\widetilde{K}_{\mathcal{I}}(\bar{X}) - \check{K}_{\mathcal{I}}(\bar{X}), \widetilde{K}_{\mathcal{I}}(X^{ai}) - \check{K}_{\mathcal{I}}(X^{ai})\right\}$$

and

$$\left|\widetilde{\pi}_{\mathrm{F}}^{bi}(c;\bar{c}) - \pi_{\mathrm{F}}^{bi}(c;\bar{c})\right| \leq \max\left\{\widetilde{K}_{\mathcal{I}}(\bar{X}) - \check{K}_{\mathcal{I}}(\bar{X}), \widetilde{K}_{\mathcal{I}}(X^{bi}) - \check{K}_{\mathcal{I}}(X^{bi})\right\}.$$

Proof. Fix any $Y, Z \in \mathcal{L}_T^2$. Observe from (5.125) that

$$0 \leq \widetilde{K}_{\mathcal{I}}(Y) - K_{\mathcal{I}}(Y) \leq \widetilde{K}_{\mathcal{I}}(Y) - \check{K}_{\mathcal{I}}(Y)$$

and

$$0 \leq \widetilde{K}_{\mathcal{I}}(Z) - K_{\mathcal{I}}(Z) \leq \widetilde{K}_{\mathcal{I}}(Z) - \check{K}_{\mathcal{I}}(Z).$$

Then

$$\begin{aligned} &\left|\left(\widetilde{K}_{\mathcal{I}}(Y) - \widetilde{K}_{\mathcal{I}}(Z)\right) - \left(K_{\mathcal{I}}(Y) - K_{\mathcal{I}}(Z)\right)\right| \\ &\qquad = \left|\left(\widetilde{K}_{\mathcal{I}}(Y) - K_{\mathcal{I}}(Y)\right) - \left(\widetilde{K}_{\mathcal{I}}(Z) - K_{\mathcal{I}}(Z)\right)\right| \\ &\qquad \leq \max\left\{\widetilde{K}_{\mathcal{I}}(Y) - K_{\mathcal{I}}(Y), \widetilde{K}_{\mathcal{I}}(Z) - K_{\mathcal{I}}(Z)\right\} \\ &\qquad \leq \max\left\{\widetilde{K}_{\mathcal{I}}(Y) - \check{K}_{\mathcal{I}}(Y), \widetilde{K}_{\mathcal{I}}(Z) - \check{K}_{\mathcal{I}}(Z)\right\}; \end{aligned}$$

the first inequality follows from $\widetilde{K}_{\mathcal{I}}(Y) - K_{\mathcal{I}}(Y) \geq 0$ and $\widetilde{K}_{\mathcal{I}}(Z) - K_{\mathcal{I}}(Z) \geq 0$. By letting $Y = \bar{X}$ and $Z = X^{ai}$, it yields

$$\begin{aligned} &\left|\left(\widetilde{K}_{\mathcal{I}}(\bar{X}) - \widetilde{K}_{\mathcal{I}}(X^{ai})\right) - \left(K_{\mathcal{I}}(\bar{X}) - K_{\mathcal{I}}(X^{ai})\right)\right| \\ &\qquad \leq \max\left\{\widetilde{K}_{\mathcal{I}}(\bar{X}) - \check{K}_{\mathcal{I}}(\bar{X}), \widetilde{K}_{\mathcal{I}}(X^{ai}) - \check{K}_{\mathcal{I}}(X^{ai})\right\}. \end{aligned}$$

Similarly, by taking $Y = X^{bi}$ and $Z = \bar{X}$, it follows that

$$\begin{aligned} &\left|\left(\widetilde{K}_{\mathcal{I}}(X^{bi}) - \widetilde{K}_{\mathcal{I}}(\bar{X})\right) - \left(K_{\mathcal{I}}(X^{bi}) - K_{\mathcal{I}}(\bar{X})\right)\right| \\ &\qquad \leq \max\left\{\widetilde{K}_{\mathcal{I}}(X^{bi}) - \check{K}_{\mathcal{I}}(X^{bi}), \widetilde{K}_{\mathcal{I}}(\bar{X}) - \check{K}_{\mathcal{I}}(\bar{X})\right\}. \end{aligned}$$

Therefore, the result follows from (5.121)-(5.122) and Theorem 5.7. $\square$

We are going to compute $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ in Examples 5.37-5.41 below. For convenience, in addition to $\mathrm{Call}_T^{\mathrm{P}}$ defined in (5.123), we define $\mathrm{Call}_T^{\mathrm{C}}$, $\mathrm{Put}_T^{\mathrm{C}}$, and $\mathrm{Put}_T^{\mathrm{P}}$ as

$$\mathrm{Call}_T^{\mathrm{C}}(A) := (\max(S_T - A, 0), 0) \tag{5.126}$$

$$\mathrm{Put}_T^{\mathrm{C}}(A) := (\max(A - S_T, 0), 0) \tag{5.127}$$

$$\mathrm{Put}_T^{\mathrm{P}}(A) := (A\mathbf{1}_{\{S_T<A\}}, -\mathbf{1}_{\{S_T<A\}}) \tag{5.128}$$

for all $A \geq 0$. The random variable $\mathrm{Call}_T^{\mathrm{C}}(A)$ (resp. $\mathrm{Put}_T^{\mathrm{C}}(A)$) represents the payoff of a European call (resp. put) option delivered by cash with strike price A; here "C" in the superscript stands for cash delivery. Similarly, the random variable $\mathrm{Put}_T^{\mathrm{P}}(A)$ is the payoff of a European put option delivered by portfolio with strike price A.

In Examples 5.37-5.39 below, we will compute the value $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ for different values of X in the models with $T = 52$. Then, in Examples 5.40-5.41, we will present $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ in the models with larger number of steps T.

Example 5.37. Let $T = 52$, $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, $\mathcal{I} = \mathcal{I}^R$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$, where $\mathcal{I}^R = \{0, \dots, T\}$ by (5.2). We calculate both $\widetilde{K}_{\mathcal{I}}(X)$ and $\check{K}_{\mathcal{I}}(X)$ for different values of X, $\widetilde{N}$, and $\check{N}$, in Table 5.2. The values $\widetilde{K}_{\mathcal{I}}(X)$ and $\check{K}_{\mathcal{I}}(X)$ appear to converge to the same value (up to 3 decimal places) as $\widetilde{N}$ and $\check{N}$ increase. However, the speed of convergence of $\widetilde{K}_{\mathcal{I}}(X)$ is faster than $\check{K}_{\mathcal{I}}(X)$. In the case when $\widetilde{N} = 20$ and $\check{N} = 300$, we have $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X) \leq 0.00302$, and this means that the error of approximating $K_{\mathcal{I}}(X)$ by $\widetilde{K}_{\mathcal{I}}(X)$ with $\widetilde{N} = 20$ will not exceed 0.00302 by (5.125). The values of $\widetilde{K}_{\mathcal{I}}(X)$ and $\check{K}_{\mathcal{I}}(X)$ in the case when $X = \mathrm{Call}_T^{\mathrm{P}}(100)$ are equal to that in the case when $X = \mathrm{Put}_T^{\mathrm{P}}(100)$. Similarly, the values of $\widetilde{K}_{\mathcal{I}}(X)$ and $\check{K}_{\mathcal{I}}(X)$ in the situation when $X = -\mathrm{Call}_T^{\mathrm{P}}(100)$ are equal to that in the situation when $X = -\mathrm{Put}_T^{\mathrm{P}}(100)$. This is because of the symmetry between $\mathrm{Call}_T^{\mathrm{P}}(100)$ and $\mathrm{Put}_T^{\mathrm{P}}(100)$, and the symmetry of the stock price movement. Similar pattern can be observed in Examples 5.38-5.41 as well, and we shall not mention it repeatedly.

Example 5.38. In this example, the values of T, r_e, p, $\mathcal{I}$, and $(\alpha_t)_{t=0}^T$ are set to be the corresponding values used in Example 5.37 above. Then we compute $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ for $k = 1\%, 2\%, 3\%$ which is higher than $k = 0.5\%$ in Example 5.37; see Table 5.3. The lower bound $\check{K}_{\mathcal{I}}(X)$ is calculated using $\check{N} = 300$. To achieve $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X) \leq 0.01$, the integer $\widetilde{N} = 30$ is already enough when $k = 1\%$. However, when $k = 2\%$ (resp. $k = 3\%$), we need $\widetilde{N}$

	$\widetilde{N} = \check{N}$					
	20	30	60	120	200	300
	$X = 0$					
$\widetilde{K}_{\mathcal{I}}(X)$	0.05749	0.05693	0.05662	0.05651	0.05647	0.05647
$\check{K}_{\mathcal{I}}(X)$	0.03120	0.04060	0.04854	0.05448	0.05612	0.05643
ϵ	0.02628	0.01633	0.00808	0.00203	0.00035	0.00004
	$X = \mathrm{Call}_T^{\mathrm{C}}(100)$					
$\widetilde{K}_{\mathcal{I}}(X)$	7.35275	7.35109	7.35014	7.34991	7.34986	7.34985
$\check{K}_{\mathcal{I}}(X)$	7.31948	7.33652	7.34630	7.34909	7.34966	7.34974
ϵ	0.03328	0.01457	0.00385	0.00083	0.00020	0.00011
	$X = -\mathrm{Call}_T^{\mathrm{C}}(100)$					
$\widetilde{K}_{\mathcal{I}}(X)$	-8.48622	-8.48739	-8.48818	-8.48835	-8.48839	-8.48841
$\check{K}_{\mathcal{I}}(X)$	-8.51621	-8.50276	-8.49202	-8.48890	-8.48852	-8.48845
ϵ	0.02999	0.01536	0.00384	0.00054	0.00013	0.00005
	$X = \mathrm{Put}_T^{\mathrm{C}}(100)$					
$\widetilde{K}_{\mathcal{I}}(X)$	7.37622	7.37454	7.37359	7.37336	7.37331	7.37329
$\check{K}_{\mathcal{I}}(X)$	7.33926	7.35772	7.36928	7.37247	7.37314	7.37323
ϵ	0.03696	0.01682	0.00431	0.00089	0.00016	0.00007
	$X = -\mathrm{Put}_T^{\mathrm{C}}(100)$					
$\widetilde{K}_{\mathcal{I}}(X)$	-8.43004	-8.43123	-8.43202	-8.43220	-8.43224	-8.43225
$\check{K}_{\mathcal{I}}(X)$	-8.45978	-8.44443	-8.43576	-8.43274	-8.43243	-8.43235
ϵ	0.02974	0.01320	0.00374	0.00054	0.00019	0.00010
	$X = \mathrm{Call}_T^{\mathrm{P}}(100)$					
$\widetilde{K}_{\mathcal{I}}(X)$	7.56499	7.56333	7.56238	7.56215	7.56210	7.56208
$\check{K}_{\mathcal{I}}(X)$	7.52811	7.54753	7.55786	7.56116	7.56184	7.56198
ϵ	0.03688	0.01581	0.00452	0.00099	0.00025	0.00010
	$X = -\mathrm{Call}_T^{\mathrm{P}}(100)$					
$\widetilde{K}_{\mathcal{I}}(X)$	-8.24621	-8.24738	-8.24817	-8.24834	-8.24838	-8.24839
$\check{K}_{\mathcal{I}}(X)$	-8.27724	-8.26181	-8.25248	-8.24903	-8.24858	-8.24848
ϵ	0.03103	0.01443	0.00432	0.00069	0.00021	0.00009
	$X = \mathrm{Put}_T^{\mathrm{P}}(100)$					
$\widetilde{K}_{\mathcal{I}}(X)$	7.56499	7.56333	7.56238	7.56215	7.56210	7.56208
$\check{K}_{\mathcal{I}}(X)$	7.52811	7.54753	7.55786	7.56116	7.56184	7.56198
ϵ	0.03688	0.01581	0.00452	0.00099	0.00025	0.00010
	$X = -\mathrm{Put}_T^{\mathrm{P}}(100)$					
$\widetilde{K}_{\mathcal{I}}(X)$	-8.24621	-8.24738	-8.24817	-8.24834	-8.24838	-8.24839
$\check{K}_{\mathcal{I}}(X)$	-8.27724	-8.26181	-8.25248	-8.24903	-8.24858	-8.24848
ϵ	0.03103	0.01443	0.00432	0.00069	0.00021	0.00009

Table 5.2: The values of $\widetilde{K}_{\mathcal{I}}(X)$, $\check{K}_{\mathcal{I}}(X)$, and $\epsilon := \widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$, where $T = 52$, $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, $\mathcal{I} = \mathcal{I}^R = \{0, \ldots, T\}$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$

k	$\check{K}_{\mathcal{I}}(X)$	$\widetilde{N}$ 30	60	120	200	300
				$X = 0$		
1%	0.01958	0.00205	0.00096	0.00056	0.00048	0.00045
2%	-0.00008	0.00167	0.00047	0.00011	0.00009	0.00008
3%	-0.00165	0.00174	0.00166	0.00165	0.00165	0.00165
				$X = \mathrm{Call}_T^{\mathrm{C}}(100)$		
1%	6.81311	0.00437	0.00138	0.00064	0.00048	0.00043
2%	5.93487	0.01602	0.00557	0.00298	0.00243	0.00226
3%	5.23887	0.03264	0.01206	0.00670	0.00550	0.00513
				$X = -\mathrm{Call}_T^{\mathrm{C}}(100)$		
1%	-9.04626	0.00296	0.00086	0.00036	0.00024	0.00021
2%	-10.03788	0.00890	0.00308	0.00169	0.00138	0.00129
3%	-10.94115	0.01510	0.00538	0.00292	0.00240	0.00224
				$X = \mathrm{Put}_T^{\mathrm{C}}(100)$		
1%	6.84863	0.00434	0.00127	0.00052	0.00036	0.00031
2%	5.97302	0.01558	0.00488	0.00220	0.00163	0.00145
3%	5.25068	0.03025	0.01007	0.00444	0.00321	0.00284
				$X = -\mathrm{Put}_T^{\mathrm{C}}(100)$		
1%	-8.92060	0.00320	0.00104	0.00052	0.00040	0.00036
2%	-9.72728	0.01007	0.00386	0.00236	0.00204	0.00194
3%	-10.37110	0.01821	0.00740	0.00465	0.00410	0.00392
				$X = \mathrm{Call}_T^{\mathrm{P}}(100)$		
1%	7.18669	0.00457	0.00152	0.00075	0.00059	0.00054
2%	6.51795	0.01882	0.00671	0.00383	0.00320	0.00300
3%	5.90147	0.04380	0.01771	0.00951	0.00808	0.00764
				$X = -\mathrm{Call}_T^{\mathrm{P}}(100)$		
1%	-8.58377	0.00320	0.00108	0.00057	0.00045	0.00041
2%	-9.18987	0.00992	0.00400	0.00257	0.00225	0.00216
3%	-9.76007	0.01795	0.00778	0.00514	0.00461	0.00444
				$X = \mathrm{Put}_T^{\mathrm{P}}(100)$		
1%	7.18669	0.00457	0.00152	0.00075	0.00059	0.00054
2%	6.51795	0.01882	0.00671	0.00383	0.00320	0.00300
3%	5.90146	0.04380	0.01771	0.00952	0.00808	0.00764
				$X = -\mathrm{Put}_T^{\mathrm{P}}(100)$		
1%	-8.58377	0.00320	0.00108	0.00057	0.00045	0.00041
2%	-9.18987	0.00992	0.00400	0.00257	0.00225	0.00215
3%	-9.76007	0.01796	0.00778	0.00514	0.00461	0.00444

Table 5.3: The value $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ ($\check{K}_{\mathcal{I}}(X)$ is calculated using $\check{N} = 300$) for various k, where $T = 52$, $r_e = 0\%$, $p = 0.5$, $\mathcal{I} = \mathcal{I}^R = \{0, \ldots, T\}$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$

p	$\check{K}_{\mathcal{I}}(X)$	$\widetilde{N}$				
		30	60	120	200	300
				$X = 0$		
0.35	56.64396	0.00104	0.00044	0.00015	0.00010	0.00008
0.55	8.04955	0.00077	0.00025	0.00012	0.00008	0.00007
0.75	203.80438	0.00199	0.00047	0.00019	0.00012	0.00010
				$X = \mathrm{Call}^{\mathrm{C}}_T(100)$		
0.35	64.30597	0.00090	0.00030	0.00009	0.00007	0.00006
0.55	15.58841	0.00184	0.00055	0.00022	0.00014	0.00012
0.75	211.51973	0.00183	0.00068	0.00028	0.00017	0.00014
				$X = -\mathrm{Call}^{\mathrm{C}}_T(100)$		
0.35	48.42610	0.00191	0.00071	0.00026	0.00017	0.00014
0.55	-0.20737	0.00133	0.00042	0.00015	0.00010	0.00008
0.75	195.62750	0.00047	0.00017	0.00008	0.00006	0.00005
				$X = \mathrm{Put}^{\mathrm{C}}_T(100)$		
0.35	64.43553	0.00092	0.00030	0.00009	0.00007	0.00006
0.55	15.58125	0.00183	0.00054	0.00021	0.00014	0.00012
0.75	211.43369	0.00183	0.00069	0.00028	0.00017	0.00014
				$X = -\mathrm{Put}^{\mathrm{C}}_T(100)$		
0.35	48.38915	0.00193	0.00071	0.00026	0.00017	0.00014
0.55	-0.11396	0.00134	0.00043	0.00016	0.00010	0.00008
0.75	195.81535	0.00048	0.00016	0.00008	0.00006	0.00005
				$X = \mathrm{Call}^{\mathrm{P}}_T(100)$		
0.35	64.53716	0.00091	0.00030	0.00009	0.00007	0.00006
0.55	15.78152	0.00182	0.00054	0.00021	0.00014	0.00011
0.75	211.64781	0.00183	0.00068	0.00028	0.00017	0.00014
				$X = -\mathrm{Call}^{\mathrm{P}}_T(100)$		
0.35	48.60124	0.00194	0.00071	0.00026	0.00017	0.00014
0.55	0.04034	0.00134	0.00042	0.00015	0.00010	0.00008
0.75	195.87862	0.00048	0.00017	0.00008	0.00006	0.00005
				$X = \mathrm{Put}^{\mathrm{P}}_T(100)$		
0.35	64.53716	0.00091	0.00030	0.00009	0.00007	0.00006
0.55	15.78152	0.00182	0.00054	0.00021	0.00014	0.00011
0.75	211.64781	0.00183	0.00068	0.00028	0.00017	0.00014
				$X = -\mathrm{Put}^{\mathrm{P}}_T(100)$		
0.35	48.60124	0.00194	0.00071	0.00026	0.00017	0.00014
0.55	0.04034	0.00134	0.00042	0.00015	0.00010	0.00008
0.75	195.87862	0.00048	0.00017	0.00008	0.00006	0.00005

Table 5.4: The value $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ ($\check{K}_{\mathcal{I}}(X)$ is calculated using $\check{N} = 300$) for various p, where $T = 52$, $r_e = 0\%$, $k = 0.5\%$, $\mathcal{I} = \mathcal{I}^R = \{0, \dots, T\}$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$

X	T	$\mathcal{I}$	$\check{K}_{\mathcal{I}}(X)$	$\widetilde{N}$		
				60	120	250
0	100	$\mathcal{I}^R$	0.10673	0.00088	0.00055	0.00046
		$\mathcal{I}^U$	0.00275	0.00003	0.00001	0.00001
	250	$\mathcal{I}^R$	0.26401	0.00476	0.00250	0.00180
		$\mathcal{I}^U$	0.00271	0.00009	0.00006	0.00005
$\mathrm{Call}^{\mathrm{C}}_T(100)$	100	$\mathcal{I}^R$	7.52083	0.00130	0.00055	0.00036
		$\mathcal{I}^U$	6.74211	0.00038	0.00028	0.00026
	250	$\mathcal{I}^R$	7.86522	0.00547	0.00214	0.00109
		$\mathcal{I}^U$	6.74144	0.00123	0.00090	0.00081
$-\mathrm{Call}^{\mathrm{C}}_T(100)$	100	$\mathcal{I}^R$	-8.34692	0.00094	0.00032	0.00016
		$\mathcal{I}^U$	-9.17638	0.00009	0.00004	0.00002
	250	$\mathcal{I}^R$	-8.04841	0.00470	0.00202	0.00117
		$\mathcal{I}^U$	-9.22013	0.00036	0.00016	0.00010
$\mathrm{Put}^{\mathrm{C}}_T(100)$	100	$\mathcal{I}^R$	7.53824	0.00122	0.00046	0.00027
		$\mathcal{I}^U$	6.77057	0.00018	0.00008	0.00005
	250	$\mathcal{I}^R$	7.86010	0.00550	0.00213	0.00108
		$\mathcal{I}^U$	6.77037	0.00070	0.00036	0.00027
$-\mathrm{Put}^{\mathrm{C}}_T(100)$	100	$\mathcal{I}^R$	-8.28506	0.00113	0.00050	0.00033
		$\mathcal{I}^U$	-9.12307	0.00027	0.00022	0.00020
	250	$\mathcal{I}^R$	-7.95383	0.00466	0.00203	0.00120
		$\mathcal{I}^U$	-9.16653	0.00083	0.00063	0.00058
$\mathrm{Call}^{\mathrm{P}}_T(100)$	100	$\mathcal{I}^R$	7.72968	0.00138	0.00062	0.00043
		$\mathcal{I}^U$	6.96042	0.00039	0.00029	0.00027
	250	$\mathcal{I}^R$	8.05636	0.00562	0.00229	0.00124
		$\mathcal{I}^U$	6.96242	0.00125	0.00092	0.00083
$-\mathrm{Call}^{\mathrm{P}}_T(100)$	100	$\mathcal{I}^R$	-8.10703	0.00113	0.00051	0.00035
		$\mathcal{I}^U$	-8.92587	0.00025	0.00020	0.00019
	250	$\mathcal{I}^R$	-7.80529	0.00491	0.00226	0.00143
		$\mathcal{I}^U$	-8.96912	0.00080	0.00061	0.00056
$\mathrm{Put}^{\mathrm{P}}_T(100)$	100	$\mathcal{I}^R$	7.72968	0.00138	0.00062	0.00043
		$\mathcal{I}^U$	6.96042	0.00039	0.00029	0.00027
	250	$\mathcal{I}^R$	8.05636	0.00562	0.00229	0.00124
		$\mathcal{I}^U$	6.96242	0.00125	0.00092	0.00083
$-\mathrm{Put}^{\mathrm{P}}_T(100)$	100	$\mathcal{I}^R$	-8.10703	0.00113	0.00051	0.00035
		$\mathcal{I}^U$	-8.92587	0.00025	0.00020	0.00019
	250	$\mathcal{I}^R$	-7.80529	0.00491	0.00227	0.00143
		$\mathcal{I}^U$	-8.96912	0.00080	0.00061	0.00056

Table 5.5: The value $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ ($\check{K}_{\mathcal{I}}(X)$ is calculated using $\check{N} = 300$) for various T and $\mathcal{I}$, where $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$

X	$\check{K}_{\mathcal{I}}(X)$	$\widetilde{N}$				
		10	20	40	60	120
0	0.00254	0.00044	0.00037	0.00032	0.00030	0.00025
$\mathrm{Call}_T^{\mathrm{C}}(100)$	6.73981	0.09711	0.02791	0.01122	0.00816	0.00632
$-\mathrm{Call}_T^{\mathrm{C}}(100)$	-9.24228	0.07325	0.01396	0.00469	0.00274	0.00152
$\mathrm{Put}_T^{\mathrm{C}}(100)$	6.76864	0.09762	0.02779	0.01095	0.00786	0.00598
$-\mathrm{Put}_T^{\mathrm{C}}(100)$	-9.18969	0.07471	0.01575	0.00648	0.00453	0.00331
$\mathrm{Call}_T^{\mathrm{P}}(100)$	6.96235	0.09830	0.02853	0.01180	0.00870	0.00682
$-\mathrm{Call}_T^{\mathrm{P}}(100)$	-8.99257	0.07385	0.01536	0.00637	0.00448	0.00330
$\mathrm{Put}_T^{\mathrm{P}}(100)$	6.96234	0.09832	0.02854	0.01181	0.00871	0.00684
$-\mathrm{Put}_T^{\mathrm{P}}(100)$	-8.99260	0.07387	0.01539	0.00639	0.00450	0.00332

Table 5.6: The value $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ ($\check{K}_{\mathcal{I}}(X)$ is calculated using $\check{N} = 120$) for $T = 1000$, where $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, $\mathcal{I} = \mathcal{I}^U = \{T\}$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$

to be at least 60 (resp. 120). As expected, in order to keep the same level of accuracy for approximating $K_{\mathcal{I}}(X)$, we should choose a higher value of $\widetilde{N}$ when k increases.

Example 5.39. We set $T = 52$, $r_e = 0\%$, $k = 0.5\%$, $\mathcal{I} = \mathcal{I}^R$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$. By considering different values of p which is the parameter forming the market probability, we present the value $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ in Table 5.4; again $\check{K}_{\mathcal{I}}(X)$ is calculated using $\check{N} = 300$. As $\widetilde{N}$ increases, the value $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ decreases and appear to converge to 0. Moreover, this difference is less than 0.002 when $\widetilde{N} = 30$ for all different values of p. In the case when $p = 0.55$, the value $\check{K}_{\mathcal{I}}(X)$ is significantly less than that when $p = 0.35, 0.75$.

Example 5.40. Let $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$. We compute the difference between $\widetilde{K}_{\mathcal{I}}(X)$ and $\check{K}_{\mathcal{I}}(X)$, for $T = 100, 250$, $\mathcal{I} = \mathcal{I}^R, \mathcal{I}^U$, $\check{N} = 300$, and various values of $\widetilde{N}$ and X, in Table 5.5; here $\mathcal{I}^U = \{T\}$ by (5.3). It shows that $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X) \leq 0.00562$ when $\widetilde{N} \geq 60$ in all cases. However, in the case when $\mathcal{I} = \mathcal{I}^U$, the value $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ is remarkably less than that when $\mathcal{I} = \mathcal{I}^R$. Similarly, in the situation when $T = 100$, the value $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ is less than that when $T = 250$. As expected, approximation error increases as the trading dates T increases. In order to maintain a certain level of accuracy for approximating $K_{\mathcal{I}}(X)$, we should set $\widetilde{N}$ to be a greater number when T increases.

Example 5.41. We take $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, $\mathcal{I} = \mathcal{I}^U$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$. Our final example is to compute $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ for $T = 1000$, $\check{N} = 120$, various values of $\widetilde{N}$ and X, and $\mathcal{I} = \mathcal{I}^U$, in Table 5.6. It suggests that the upper bound of the approximation error $\widetilde{K}_{\mathcal{I}}(X) - \check{K}_{\mathcal{I}}(X)$ will not exceed 0.00871 when $\widetilde{N} \geq 60$.

5.5.2 Optimal injections and minimal regret

In this section, we will present the numerical solution to the problem (3.19) with the sequence of regret functions defined in (5.1) in a number of examples.

Let $X = -\sum_{t=0}^{T} u_t$, where $u = (u_t)_{t=0}^{T} \in \mathcal{N}^2$ is given and u represents the investor's liability. We are going to introduce a method to approximate the minimal regret $V(u)$; see Theorem 5.5 for a formula of $V(u)$. Let

$$\widetilde{\lambda}(u) := \exp\left[\frac{1}{\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t}}\left(\sum_{t\in\mathcal{I}}\frac{\ln\alpha_t}{\alpha_t} - \widetilde{K}_{\mathcal{I}}(X)\right)\right]$$
$$\widetilde{V}(u) := \widetilde{\lambda}(u)\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t} - |\mathcal{I}|$$

(cf. (5.11) and Theorem 5.5). Observe that $\widetilde{\lambda}(u)$ depends on $\widetilde{K}_{\mathcal{I}}(X)$, and hence it depends on $\widetilde{N}$; see (5.119) and the comments preceding it. Combining (5.119) and (5.11), we have

$$\lim_{\widetilde{N}\to\infty}\widetilde{\lambda}(u) = \hat{\lambda}(u),$$

where $\hat{\lambda}(u)$ defined in (5.11) is the unique solution to the problem (5.10); see Proposition 5.4. This means

$$\lim_{\widetilde{N}\to\infty}\widetilde{V}(u) = \hat{\lambda}(u)\sum_{t\in\mathcal{I}}\frac{1}{\alpha_t} - |\mathcal{I}| = V(u)$$

(Theorem 5.5). We will use $\widetilde{\lambda}(u)$ and $\widetilde{V}(u)$ to approximate $\hat{\lambda}(u)$ and $V(u)$ respectively. This completes the construction of the approximation of $V(u)$.

Based on $(J_t)_{t=0}^{T}$, a pair $(\hat{\mathbb{Q}}, \hat{S})$ is constructed in Algorithm 5.19, where $(\hat{\mathbb{Q}}, \hat{S}) \in \mathcal{P}$ is a solution to the problem (5.7); see Theorem 5.20. By approximating $(J_t)_{t=0}^{T}$ by $(\widetilde{J}_t)_{t=0}^{T}$ in this algorithm (i.e. using $(\widetilde{J}_t)_{t=0}^{T}$ instead of $(J_t)_{t=0}^{T}$), we can construct a pair $(\widetilde{\mathbb{Q}}, \widetilde{S})$ based on $(\widetilde{J}_t)_{t=0}^{T}$ to approximate $(\hat{\mathbb{Q}}, \hat{S})$. One can show that $(\widetilde{\mathbb{Q}}, \widetilde{S}) \in \bar{\mathcal{P}}$ by using a similar method in proving $(\hat{\mathbb{Q}}, \hat{S}) \in \bar{\mathcal{P}}$ in the beginning of the proof of Theorem 5.20 (see p. 164). In the situation when $(\widetilde{\mathbb{Q}}, \widetilde{S}) \in \mathcal{P}$ (i.e. $\widetilde{\mathbb{Q}}(\omega) > 0$ for all $\omega \in \Omega$), we define $(\widetilde{x}_t)_{t=0}^{T} \in \mathcal{N}$ as

$$\widetilde{x}_t := \begin{cases} \frac{1}{\alpha_t}\ln\frac{\widetilde{\lambda}(u)\Lambda_t^{\widetilde{\mathbb{Q}}}}{\alpha_t} & \text{if } t \in \mathcal{I} \\ 0 & \text{if } t \in \{0,\dots,T\}\setminus\mathcal{I} \end{cases} \tag{5.129}$$

(cf. Theorem 5.6). In all numerical examples presented in this section, we always have $(\widetilde{\mathbb{Q}}, \widetilde{S}) \in \mathcal{P}$. We will use $(\widetilde{x}_t)_{t=0}^{T}$ to appproximate $(\hat{x}_t)_{t=0}^{T}$ constructed in Theorem 5.6; the process $(\hat{x}_t)_{t=0}^{T}$ represents the optimal injection

	Injection $\widetilde{x}_t^{\nu_t}$			Increment $\Delta\widetilde{x}_t^{\nu_t}$	
Node	Time step			Time step	
$\nu \in \Omega_2$	$t=0$	$t=1$	$t=2$	$t=1$	$t=2$
		$k=0\%$		$k=0\%$	
uu	2.35061	2.27740	2.20419	-0.07321	-0.07321
ud	2.35061	2.27740	2.34561	-0.07321	0.06821
du	2.35061	2.41882	2.34561	0.06821	-0.07321
dd	2.35061	2.41882	2.48703	0.06821	0.06821
		$k=0.5\%$		$k=0.5\%$	
uu	2.52366	2.44787	2.44810	-0.07580	0.00024
ud	2.52366	2.44787	2.44763	-0.07580	-0.00024
du	2.52366	2.59412	2.52091	0.07045	-0.07321
dd	2.52366	2.59412	2.66233	0.07045	0.06821
		$k=3\%$		$k=3\%$	
uu	3.39788	3.31135	3.62275	-0.08653	0.31140
ud	3.39788	3.31135	2.85669	-0.08653	-0.45466
du	3.39788	3.47751	3.40431	0.07964	-0.07321
dd	3.39788	3.47751	3.54573	0.07964	0.06821

Table 5.7: Optimal injection $(\widetilde{x}_t)_{t=0}^2$ and injection increment $(\Delta\widetilde{x}_t)_{t=1}^2$ for various values of k, where $\widetilde{N}=90$, $\sum_{t=0}^2 u_t = \mathrm{Call}_2^{\mathrm{C}}(100)$, $r_e=0\%$, $p=0.5$, $\mathcal{I}=\{0,1,2\}$, and $\alpha_t=1$ for all $t\in\mathcal{I}$

at each time step and it solves the problem (3.19) with the sequence of regret functions defined in (5.1).

Notice that, as long as $\sum_{t=0}^T u_t$ is given (without knowing $(u_t)_{t=0}^T$), we can compute $X=-\sum_{t=0}^T u_t$ and then compute $\widetilde{\lambda}(u)$, $\widetilde{V}(u)$, and $(\widetilde{x}_t)_{t=0}^T$. Thus, in the examples below, we will directly specify $\sum_{t=0}^T u_t$ instead of defining $(u_t)_{t=0}^T$. In Examples 5.42-5.45, we will present the optimal injection $(\widetilde{x}_t)_{t=0}^2$ in a model with $T=2$ by varying parameters k, p, $(\alpha_t)_{t\in\mathcal{I}}$, and $\mathcal{I}$ respectively and keeping other parameter values fixed; the integer $\widetilde{N}$ will set to be 90. In particular, the injection increment $(\Delta\widetilde{x}_t)_{t=1}^2=(\widetilde{x}_t-\widetilde{x}_{t-1})_{t=1}^2$ will also be provided in Examples 5.42-5.44.

Example 5.42. Let $r_e=0\%$, $p=0.5$, $\mathcal{I}=\{0,1,2\}$, and $\alpha_t=1$ for all $t\in\mathcal{I}$. Then we compute $(\widetilde{x}_t)_{t=0}^2$ and $(\Delta\widetilde{x}_t)_{t=1}^2$ with $\sum_{t=0}^2 u_t=\mathrm{Call}_2^{\mathrm{C}}(100)$ in the models with $k=0\%, 0.5\%, 3\%$ respectively in Table 5.7. Notice that the process $(\widetilde{x}_t)_{t=0}^2$ is recombinant only when $k=0\%$. In addition, the size of cash injection $(\widetilde{x}_t)_{t=0}^2$ tends to be larger when k is higher. By straightforward calculation, we have $\mathrm{Call}_2^{\mathrm{C}}(100)=0$ on $d=\{dd,du\}$, and the increment of cash injection $\Delta\widetilde{x}_2$ on d is the same for all different k. The minimal regret $V(u)$ in this example is given by 28.47585, 34.42259, 86.70179 respectively for $k=0\%, 0.5\%, 3\%$.

	Injection $\widetilde{x}_t^{\nu_t}$			Increment $\Delta\widetilde{x}_t^{\nu_t}$	
Node	Time step			Time step	
$\nu \in \Omega_2$	$t=0$	$t=1$	$t=2$	$t=1$	$t=2$
		$p=0.2$		$p=0.2$	
uu	2.30467	3.14516	4.02171	0.84050	0.87654
ud	2.30467	3.14516	2.71339	0.84050	-0.43178
du	2.30467	1.90512	2.66975	-0.39955	0.76463
dd	2.30467	1.90512	1.56677	-0.39955	-0.33835
		$p=0.5$		$p=0.5$	
uu	2.47380	2.39800	2.35825	-0.07580	-0.03975
ud	2.47380	2.39800	2.43623	-0.07580	0.03823
du	2.47380	2.54425	2.47104	0.07045	-0.07321
dd	2.47380	2.54425	2.61246	0.07045	0.06821
		$p=0.8$		$p=0.8$	
uu	2.20680	1.66100	1.19123	-0.54580	-0.46977
ud	2.20680	1.66100	2.57705	-0.54580	0.91605
du	2.20680	3.19354	2.65033	0.98674	-0.54321
dd	2.20680	3.19354	4.17805	0.98674	0.98450

Table 5.8: Optimal injection $(\widetilde{x}_t)_{t=0}^2$ and injection increment $(\Delta\widetilde{x}_t)_{t=1}^2$ for various values of p, where $\widetilde{N}=90$, $\sum_{t=0}^2 u_t = \mathrm{Call}_2^{\mathrm{P}}(100)$, $r_e=0\%$, $k=0.5\%$, $\mathcal{I}=\{0,1,2\}$, and $\alpha_t=1$ for all $t\in\mathcal{I}$

Example 5.43. In Table 5.8, we present the processes $(\widetilde{x}_t)_{t=0}^2$ and $(\Delta\widetilde{x}_t)_{t=1}^2$ with $\sum_{t=0}^2 u_t = \mathrm{Call}_2^{\mathrm{P}}(100)$ for $p=0.2,0.5,0.8$, where $r_e=0\%$, $k=0.5\%$, $\mathcal{I}=\{0,1,2\}$, and $\alpha_t=1$ for all $t\in\mathcal{I}$. The higher value of p leads to higher values of $\Delta\widetilde{x}_1^d$, $\Delta\widetilde{x}_2^{ud}$, and $\Delta\widetilde{x}_2^{dd}$, but leads to lower values of $\Delta\widetilde{x}_1^u$, $\Delta\widetilde{x}_2^{uu}$, and $\Delta\widetilde{x}_2^{du}$. When $p=0.2,0.8$, the size of increments $|\Delta\widetilde{x}_1|$ and $|\Delta\widetilde{x}_2|$ have higher values compared to that when $p=0.5$. In the situation when $p=0.2,0.5,0.8$, the minimal regret $V(u)$ is 27.06257, 32.60221, 24.25975 respectively.

Example 5.44. The quantity α_t characterises the investor's risk preference at time $t\in\mathcal{I}$. Higher value of α_t represents higher level of risk aversion. Table 5.9 computes the optimal injection and the injection increment by considering four different $(\alpha_t)_{t\in\mathcal{I}}$. We set $r_e=0\%$, $p=0.5$, $k=0.5\%$, $\mathcal{I}=\{0,1,2\}$, and $\sum_{t=0}^2 u_t = -\mathrm{Put}_2^{\mathrm{C}}(100)$ in this example. In the case when $(\alpha_t)_{t\in\mathcal{I}}=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, the size of increment of injections $|\Delta\widetilde{x}_1|$ and $|\Delta\widetilde{x}_2|$ are higher than that when $(\alpha_t)_{t\in\mathcal{I}}=(1,1,1)$. This is because the investor is less risk averse in the situation when $(\alpha_t)_{t\in\mathcal{I}}=(\frac{1}{2},\frac{1}{2},\frac{1}{2})$. The process $(\alpha_t)_{t\in\mathcal{I}}=(\frac{3}{2},1,\frac{1}{2})$ models decreasing risk aversion, and it suggests that the cash amount of withdrawal $-\widetilde{x}_t$ is increasing in t. Similarly, the process $(\alpha_t)_{t\in\mathcal{I}}=(\frac{1}{2},1,\frac{3}{2})$ models increasing risk aversion. In such case, the optimal withdraw $-\widetilde{x}_t$ is decreasing in t. Last but not the least, while $(\alpha_t)_{t\in\mathcal{I}}=(\frac{1}{2},\frac{1}{2},\frac{1}{2}),(1,1,1),(\frac{3}{2},1,\frac{1}{2}),(\frac{1}{2},1,\frac{3}{2})$, the minimal

	Injection $\widetilde{x}_t^{\nu_t}$			Increment $\Delta\widetilde{x}_t^{\nu_t}$	
Node	Time step			Time step	
$\nu \in \Omega_2$	$t=0$	$t=1$	$t=2$	$t=1$	$t=2$
	$(\alpha_t)_{t=0}^2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$			$(\alpha_t)_{t=0}^2 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$	
uu	-2.19599	-2.33686	-2.48328	-0.14088	-0.14642
ud	-2.19599	-2.33686	-2.20044	-0.14088	0.13643
du	-2.19599	-2.06439	-2.06392	0.13160	0.00047
dd	-2.19599	-2.06439	-2.06486	0.13160	-0.00047
	$(\alpha_t)_{t=0}^2 = (1, 1, 1)$			$(\alpha_t)_{t=0}^2 = (1, 1, 1)$	
uu	-2.19406	-2.26450	-2.33771	-0.07044	-0.07321
ud	-2.19406	-2.26450	-2.19628	-0.07044	0.06821
du	-2.19406	-2.12826	-2.12802	0.06580	0.00024
dd	-2.19406	-2.12826	-2.12850	0.06580	-0.00024
	$(\alpha_t)_{t=0}^2 = (\frac{3}{2}, 1, \frac{1}{2})$			$(\alpha_t)_{t=0}^2 = (\frac{3}{2}, 1, \frac{1}{2})$	
uu	-1.67061	-2.17088	-3.10189	-0.50028	-0.93101
ud	-1.67061	-2.17088	-2.81905	-0.50028	-0.64816
du	-1.67061	-2.03465	-2.68253	-0.36404	-0.64788
dd	-1.67061	-2.03465	-2.68347	-0.36404	-0.64883
	$(\alpha_t)_{t=0}^2 = (\frac{1}{2}, 1, \frac{3}{2})$			$(\alpha_t)_{t=0}^2 = (\frac{1}{2}, 1, \frac{3}{2})$	
uu	-2.81207	-2.16962	-1.76553	0.64245	0.40409
ud	-2.81207	-2.16962	-1.67125	0.64245	0.49837
du	-2.81207	-2.03338	-1.62574	0.77869	0.40764
dd	-2.81207	-2.03338	-1.62606	0.77869	0.40733

Table 5.9: Optimal injection $(\widetilde{x}_t)_{t=0}^2$ and injection increment $(\Delta\widetilde{x}_t)_{t=1}^2$ for various $(\alpha_t)_{t\in\mathcal{I}}$, where $\widetilde{N} = 90$, $\sum_{t=0}^2 u_t = -\mathrm{Put}_2^{\mathrm{C}}(100)$, $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, and $\mathcal{I} = \{0, 1, 2\}$

	Injection $\widetilde{x}_t^{\nu_t}$			Total injection $\sum_{t=0}^{2} \widetilde{x}_t^{\nu_t}$
Node	Time step			
$\nu \in \Omega_2$	$t = 0$	$t = 1$	$t = 2$	
	$\mathcal{I} = \{0, 1, 2\}$			$\mathcal{I} = \{0, 1, 2\}$
uu	-2.22952	-2.37039	-2.51681	-7.11672
ud	-2.22952	-2.37039	-2.23397	-6.83388
du	-2.22952	-2.09792	-2.09744	-6.42488
dd	-2.22952	-2.09792	-2.09839	-6.42583
	$\mathcal{I} = \{1, 2\}$			$\mathcal{I} = \{1, 2\}$
uu	0	-3.48515	-3.63157	-7.11672
ud	0	-3.48515	-3.34873	-6.83388
du	0	-3.21268	-3.21220	-6.42488
dd	0	-3.21268	-3.21315	-6.42583
	$\mathcal{I} = \{0, 2\}$			$\mathcal{I} = \{0, 2\}$
uu	-3.34196	0	-3.62926	-6.97122
ud	-3.34196	0	-3.34641	-6.68838
du	-3.34196	0	-3.20989	-6.55185
dd	-3.34196	0	-3.21084	-6.55280
	$\mathcal{I} = \{2\}$			$\mathcal{I} = \{2\}$
uu	0	0	-6.97122	-6.97122
ud	0	0	-6.68838	-6.68838
du	0	0	-6.55185	-6.55185
dd	0	0	-6.55280	-6.55280

Table 5.10: Optimal injection $(\widetilde{x}_t)_{t=0}^{2}$ and its sum $\sum_{t=0}^{2} \widetilde{x}_t$ for various $\mathcal{I}$, where $\widetilde{N} = 90$, $\sum_{t=0}^{2} u_t = -\mathrm{Put}_2^{\mathrm{C}}(100)$, $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$

regret $V(u)$ is given by -1.99938, -2.66561, -2.55119, -2.55063 respectively.

Example 5.45. In this example, we set $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, $\alpha_t = 1$ for all $t \in \mathcal{I}$, and

$$\sum_{t=0}^{2} u_t = -\mathrm{Put}_2^{\mathrm{C}}(100).$$

In Table 5.10, we provide the optimal injection and total injection by considering four different $\mathcal{I}$: $\{0, 1, 2\}$, $\{1, 2\}$, $\{0, 2\}$, and $\{2\}$. It shows that $\widetilde{x}_t = 0$ whenever $t \notin \mathcal{I}$. Moreover, the total injection is the same for $\mathcal{I} = \{0, 1, 2\}$ and $\mathcal{I} = \{1, 2\}$. Similarly, the total injection is also the same for $\mathcal{I} = \{0, 2\}$ and $\mathcal{I} = \{2\}$. The minimal regret $V(u)$ is given by -2.55063, -1.62431, -1.62388, -0.96463 respectively for $\mathcal{I} = \{0, 1, 2\}, \{1, 2\}, \{0, 2\}, \{2\}$.

In Examples 5.46-5.47 below, we will compute the optimal injections in the market model with $T = 52$; the integer $\widetilde{N}$ is set to be 60. In these two examples, we will set $r_e = 0\%$ and $k = 0.5\%$, which ensures $(\widetilde{\mathbb{Q}}, \widetilde{S}) \in \mathcal{P}$. Indeed, by straightforward calculation, it follows that 1+u = 1.02812, 1+r = 1, and $1+\mathrm{d}=0.97265$. Moreover, for each $t=0,\dots,T-1$ and $\nu\in\Omega_t$, we have from the values of u, r, d, and k that

$$S^{a\nu d}_{t+1}<S^{b\nu}_t,\qquad S^{a\nu}_t<S^{b\nu u}_{t+1},$$

which means that there is no overlap among the intervals

$$\left[S^{b\nu d}_{t+1},S^{a\nu d}_{t+1}\right],\quad \left[S^{b\nu}_{t},S^{a\nu}_{t}\right],\quad \left[S^{b\nu u}_{t+1},S^{a\nu u}_{t+1}\right].$$

Then it follows from $S^b\le\widetilde{S}\le S^a$ (because $(\widetilde{\mathbb{Q}},\widetilde{S})\in\bar{\mathcal{P}}$) that

$$\widetilde{S}^{\nu d}_{t+1}<\widetilde{S}^{\nu}_t<\widetilde{S}^{\nu u}_{t+1}.$$

Combining this with the fact that $\widetilde{S}$ is a $\widetilde{\mathbb{Q}}$-martingale, the transition probabilities of $\widetilde{\mathbb{Q}}$ must take their values in $(0,1)$. This means that $\widetilde{\mathbb{Q}}(\omega)>0$ for all $\omega\in\Omega$ and hence $(\widetilde{\mathbb{Q}},\widetilde{S})\in\mathcal{P}$. Therefore, the optimal injection $(\widetilde{x}_t)_{t=0}^{52}$ in (5.129) is well defined in Examples 5.46-5.47 below.

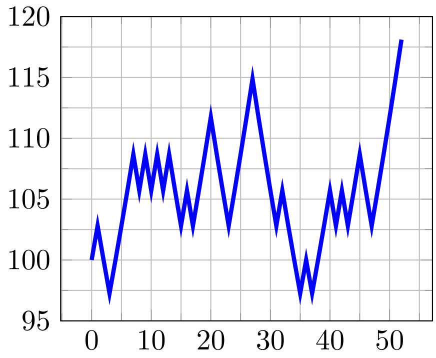
Example 5.46. Firstly, we set $T=52$, $r_e=0\%$, $p=0.5$, $k=0.5\%$, $\mathcal{I}=\mathcal{I}^R=\{0,\dots,T\}$, $\alpha_t=1$ for all $t\in\mathcal{I}$, and $\sum_{t=0}^{T}u_t=\mathrm{Call}^{\mathrm{C}}_T(100)$. On the path presented in Figure 5.7(a), we compute $(\widetilde{x}_t)_{t=0}^{52}$ and $(\Delta\widetilde{x}_t)_{t=1}^{52}$ in Figures 5.7(b) and 5.7(c) respectively. Moreover, we provide $(\widetilde{q}_t)_{t=1}^{52}$ and $\widetilde{S}=(\widetilde{S}_t)_{t=0}^{52}$ along this path in Figures 5.7(d) and 5.7(e) respectively, where $(\widetilde{q}_t)_{t=1}^{52}$ is the transition probabilities of $\widetilde{\mathbb{Q}}$; see the comments preceding (5.129) for the definition of $(\widetilde{\mathbb{Q}},\widetilde{S})$. Observe from (5.129) together with $\alpha_t=1$ and $p_t=p=0.5$ that for all $t=1,\dots,52$ the increment of injections $\Delta\widetilde{x}_t$ can be written as

$$\Delta\widetilde{x}_t=\ln\widetilde{\lambda}(u)\Lambda^{\widetilde{\mathbb{Q}}}_t-\ln\widetilde{\lambda}(u)\Lambda^{\widetilde{\mathbb{Q}}}_{t-1}=\ln\frac{\widetilde{q}_t}{p_t}=\ln\frac{\widetilde{q}_t}{0.5}.$$

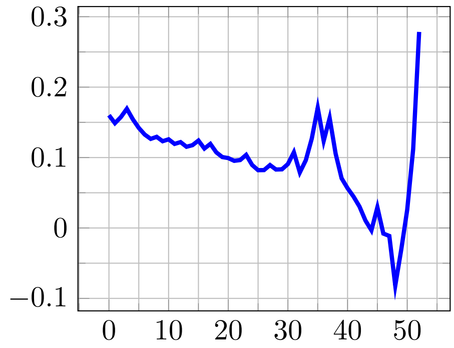
This gives a link between $(\Delta\widetilde{x}_t)_{t=1}^{52}$ and $(\widetilde{q}_t)_{t=1}^{52}$; see Figures 5.7(c) and 5.7(d). In Figure 5.7(f), the following $\{-1,0,1\}$-valued $\mathcal{F}_t$-measurable random variable

$$\widetilde{\psi}_t:=\mathbf{1}_{\left\{\widetilde{S}_t=S^a_t\right\}}-\mathbf{1}_{\left\{\widetilde{S}_t=S^b_t\right\}}=\begin{cases}1 & \text{on }\left\{S^b_t<\widetilde{S}_t=S^a_t\right\}\\ 0 & \text{on }\left\{S^b_t=\widetilde{S}_t=S^a_t\right\}\bigcup\left\{S^b_t<\widetilde{S}_t<S^a_t\right\}\\ -1 & \text{on }\left\{S^b_t=\widetilde{S}_t<S^a_t\right\}\end{cases}$$

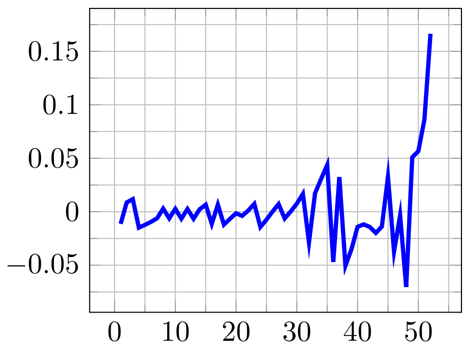
is used to indicate the position of $\widetilde{S}_t$ in $[S^b_t,S^a_t]$ at each time step t on the given path. It shows that $\widetilde{S}_t$ can be at the boundary of $[S^b_t,S^a_t]$, and it can also be in the interior of $[S^b_t,S^a_t]$.



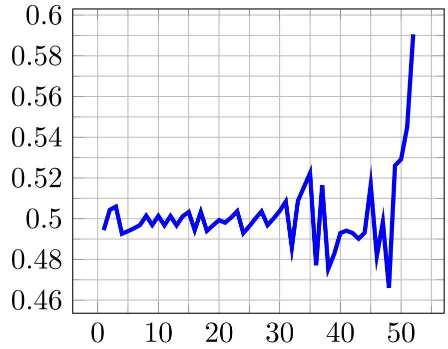
(a) Path $\left(S_t^{\nu_t}\right)_{t=0}^{52} = \left(\bar{S}_t^{\nu_t}\right)_{t=0}^{52}$



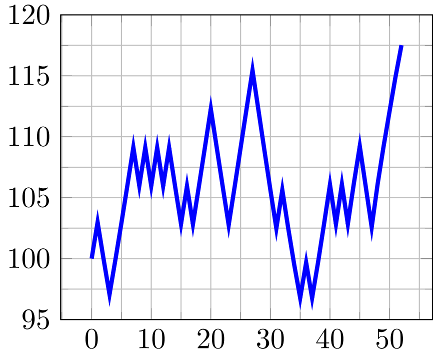
(b) Optimal injection $\left(\widetilde{x}_t^{\nu_t}\right)_{t=0}^{52}$



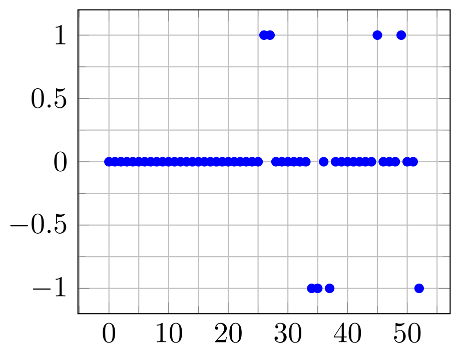
(c) Injection increment $\left(\Delta\widetilde{x}_t^{\nu_t}\right)_{t=1}^{52}$



(d) $\left(\widetilde{q}_t^{\nu_t}\right)_{t=1}^{52}$

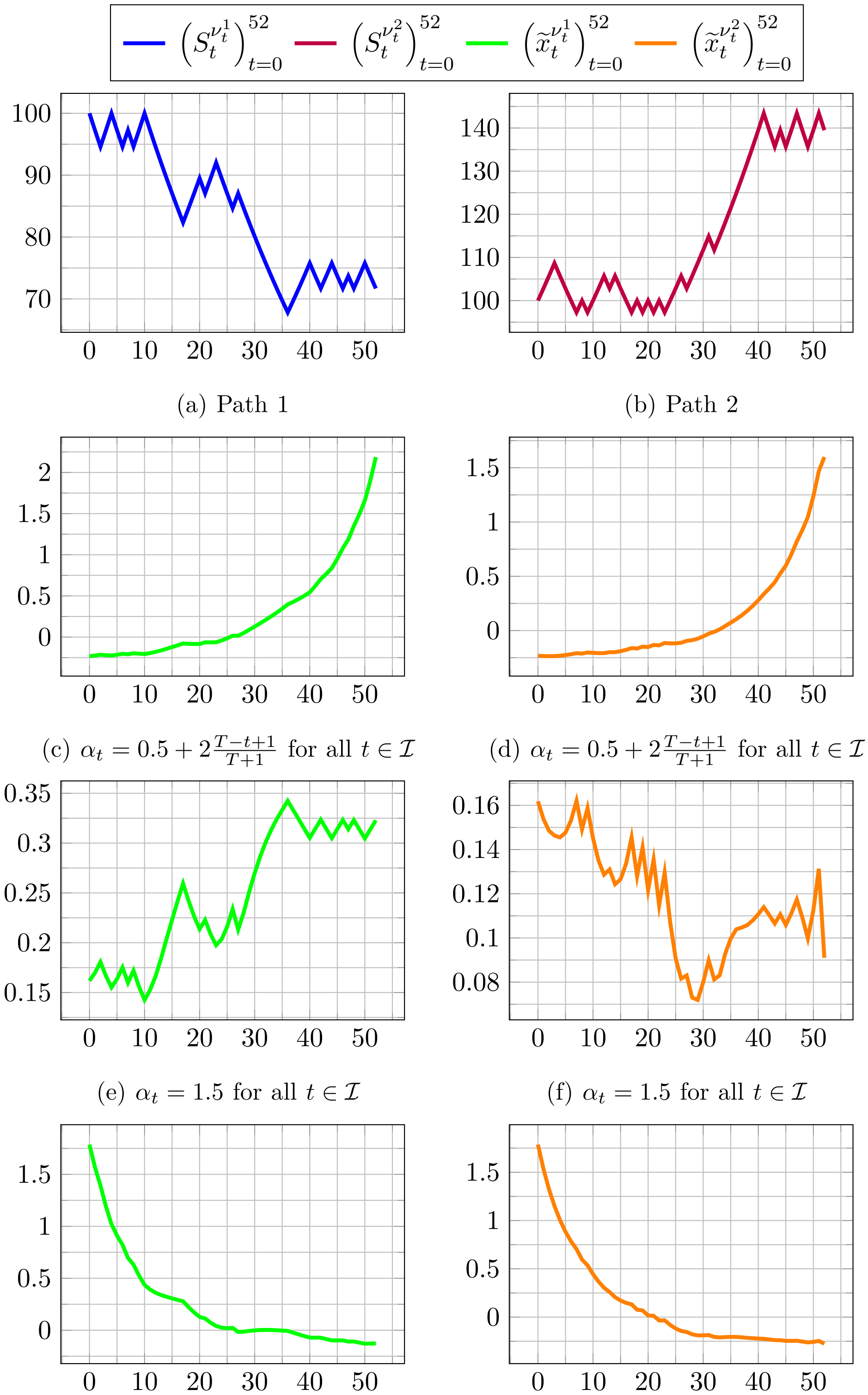


(e) $\left(\widetilde{S}_t^{\nu_t}\right)_{t=0}^{52}$



(f) $\left(\widetilde{\psi}_t^{\nu_t}\right)_{t=0}^{52}$

Figure 5.7: Optimal injection $(\widetilde{x}_t)_{t=0}^{T}$ and the pair $(\widetilde{\mathbb{Q}}, \widetilde{S})$, where $\widetilde{N} = 60$, $\sum_{t=0}^{T} u_t = \mathrm{Call}_T^{\mathrm{C}}(100)$, $T = 52$, $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, $\mathcal{I} = \mathcal{I}^R$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$



(a) Path 1
(b) Path 2
(c) $\alpha_t = 0.5 + 2\frac{T-t+1}{T+1}$ for all $t \in \mathcal{I}$
(d) $\alpha_t = 0.5 + 2\frac{T-t+1}{T+1}$ for all $t \in \mathcal{I}$
(e) $\alpha_t = 1.5$ for all $t \in \mathcal{I}$
(f) $\alpha_t = 1.5$ for all $t \in \mathcal{I}$
(g) $\alpha_t = 0.5 + 2\frac{t+1}{T+1}$ for all $t \in \mathcal{I}$
(h) $\alpha_t = 0.5 + 2\frac{t+1}{T+1}$ for all $t \in \mathcal{I}$

Figure 5.8: Optimal injection on two different paths for various $(\alpha_t)_{t\in\mathcal{I}}$, where $\sum_{t=0}^{T} u_t = \mathrm{Call}_T^{\mathrm{C}}(100)$, $\widetilde{N} = 60$, $T = 52$, $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, and $\mathcal{I} = \mathcal{I}^R$

Example 5.47. In this example, we set $T = 52$, $r_e = 0\%$, $p = 0.5$, $k = 0.5\%$, $\mathcal{I} = \mathcal{I}^R = \{0, \dots, T\}$, and $\sum_{t=0}^{T} u_t = \mathrm{Call}_T^{\mathrm{C}}(100)$. In Figure 5.8, we consider the optimal injection on two paths for three different $(\alpha_t)_{t \in \mathcal{I}}$. In the case when $t \mapsto \alpha_t$ is decreasing, there is a general upward trend in the optimal cash injection with small fluctuation over time; see Figures 5.8(c) and 5.8(d). Similarly, in the case when $t \mapsto \alpha_t$ is increasing, the optimal cash injection declines gradually in general and has a few minor fluctuation during some time periods; see Figures 5.8(g) and 5.8(h). While α_t is constant in t, there is no upward or downward trend in the cash injection process over time; see Figures 5.8(e) and 5.8(f).

5.5.3 Regret indifference prices

In this section, we will present numerical results for regret indifference prices. Firstly, in Examples 5.48-5.53, we will present the indifference prices in models with 52 time steps. These examples are used to study the influence of different parameter values on prices. Then Example 5.54 will provide the indifference prices of various options in both 250-step and 1000-step models.

We define the payoffs of a *strangle* and a *butterfly,* which depend on two parameters $A^1 \leq A^2$, as follows:

$$\begin{aligned} \mathrm{Str}_T^{\mathrm{C}}(A^1, A^2) &:= \mathrm{Put}_T^{\mathrm{C}}(A^1) + \mathrm{Call}_T^{\mathrm{C}}(A^2), \\ \mathrm{But}_T^{\mathrm{C}}(A^1, A^2) &:= \mathrm{Call}_T^{\mathrm{C}}(A^1) + \mathrm{Call}_T^{\mathrm{C}}(A^2) - 2\mathrm{Call}_T^{\mathrm{C}}(\tfrac{A^1+A^2}{2}); \end{aligned}$$

the integer T is the number of time steps in the market model, and "C" in the superscript stands for cash delivery. Notice that the payoff of a strangle can be written as

$$\mathrm{Str}_T^{\mathrm{C}}(A^1, A^2) = \begin{cases} A^1 - S_T & \text{if } S_T \leq A^1, \\ 0 & \text{if } A^1 < S_T \leq A^2, \\ S_T - A^2 & \text{if } A^2 < S_T. \end{cases}$$

Similarly, the payoff of a butterfly can be written as

$$\mathrm{But}_T^{\mathrm{C}}(A^1, A^2) = \begin{cases} S_T - A^1 & \text{if } A^1 < S_T \leq \frac{A^1+A^2}{2}, \\ A^2 - S_T & \text{if } \frac{A^1+A^2}{2} < S_T \leq A^2, \\ 0 & \text{otherwise.} \end{cases}$$

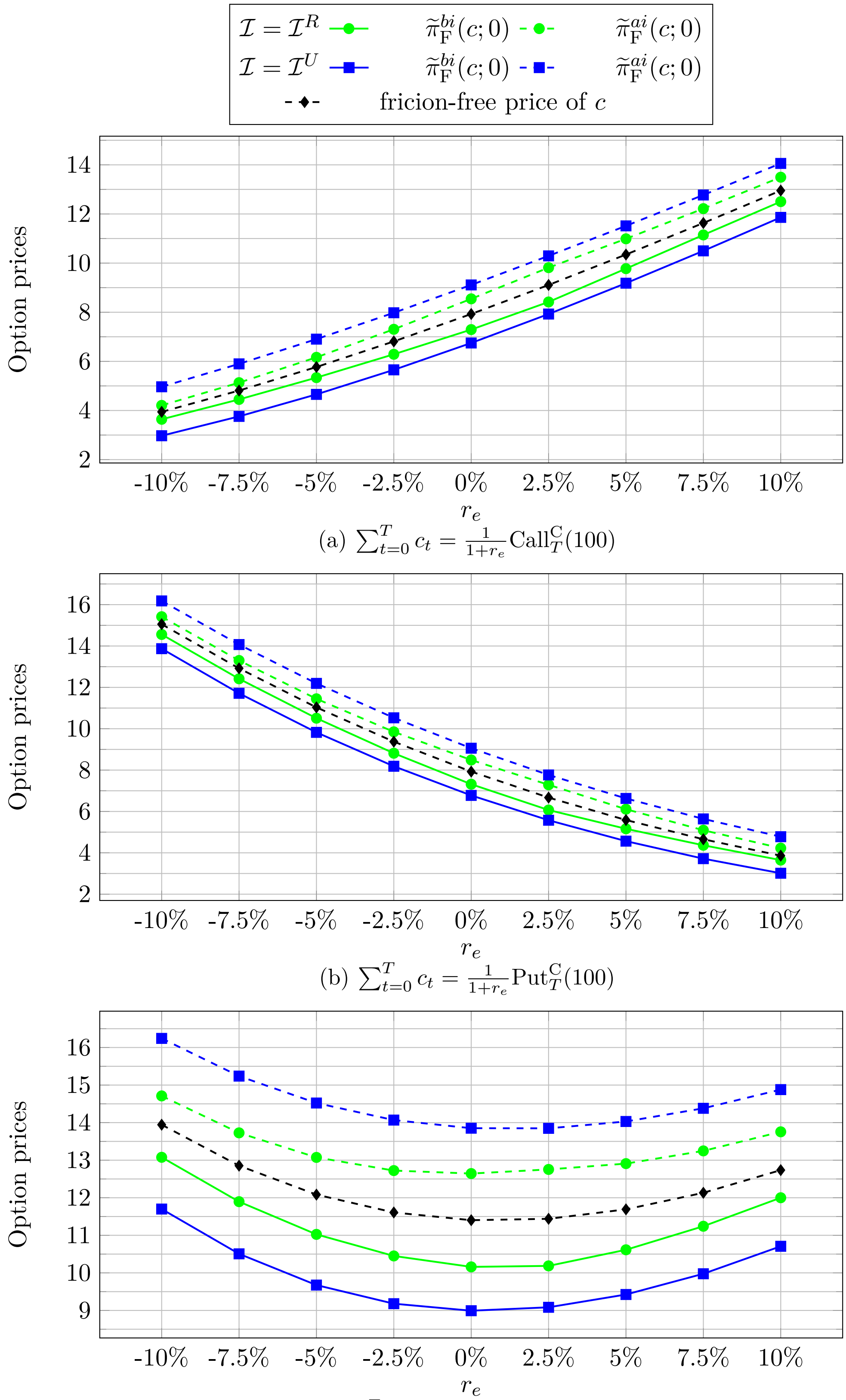
Let $c = (c_t)_{t=0}^T, \bar{c} = (\bar{c}_t)_{t=0}^T \in \mathcal{N}^2$. In Examples 5.48-5.54 below, we shall always use $\widetilde{\pi}_{\mathrm{F}}^{ai}(c; \bar{c})$ and $\widetilde{\pi}_{\mathrm{F}}^{bi}(c; \bar{c})$ defined in (5.121)-(5.122) to approximate the seller's and buyer's indifference prices $\pi_{\mathrm{F}}^{ai}(c;\bar{c})$ and $\pi_{\mathrm{F}}^{bi}(c;\bar{c})$ respectively. From the comments following (5.122), the values $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;\bar{c})$ and $\widetilde{\pi}_{\mathrm{F}}^{bi}(c;\bar{c})$ only relies on $\sum_{t=0}^{T} c_t$ and $\sum_{t=0}^{T} \bar{c}_t$. Thus, to compute $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;\bar{c})$ and $\widetilde{\pi}_{\mathrm{F}}^{bi}(c;\bar{c})$, it is enough to know $\sum_{t=0}^{T} c_t$ and $\sum_{t=0}^{T} \bar{c}_t$. In the calculation of $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;\bar{c})$ and $\widetilde{\pi}_{\mathrm{F}}^{bi}(c;\bar{c})$, the integer $\widetilde{N}$ will set to be 60; Theorem 5.36 together with Examples 5.37-5.41 suggests that the approximation error of indifference prices should be less than 0.01 under $\widetilde{N} = 60$ in most situations. We will compute the indifference prices for both $\mathcal{I} = \mathcal{I}^R$ and $\mathcal{I} = \mathcal{I}^U$ in each examples, where $\mathcal{I}^R$ and $\mathcal{I}^U$ are defined in (5.2)-(5.3). In the numerical results, the buyer's indifference price $\widetilde{\pi}_{\mathrm{F}}^{bi}(c;\bar{c})$ will always be dominated by the seller's indifference price $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;\bar{c})$.

In Examples 5.48-5.53 below, we are going to present indifference prices by varying r_e, p, $\bar{c}$, option strike prices, k, and $(\alpha_t)_{t=0}^{T}$ respectively and keep other parameters fixed. In these examples, we set $T = 52$.

Example 5.48. Firstly, let $p = 0.5$, $k = 0.5\%$, $\alpha_t = 1$ for all $t \in \mathcal{I}$, and $\bar{c}_t = 0$ for all $t = 0, \dots, T$. We consider the indifference prices of c for various values of r_e which is the annually compounded interest rate. In Figure 5.9(a), we set $\sum_{t=0}^{T} c_t = \frac{1}{1+r_e}\mathrm{Call}_T^{\mathrm{C}}(100)$ to be the discounted payoff of a call option. It shows that as r_e increases, the price of c in the friction-free model (i.e. $k = 0\%$) increases, and all indifference prices increase as well. Similarly, in Figure 5.9(b), we set $\sum_{t=0}^{T} c_t = \frac{1}{1+r_e}\mathrm{Put}_T^{\mathrm{C}}(100)$ which is the discounted payoff of a put option. It shows that an increase in r_e leads to a decrease in all prices. In Figure 5.9(c), we set $\sum_{t=0}^{T} c_t = \frac{1}{1+r_e}\mathrm{Str}_T^{\mathrm{C}}(95, 105)$ which is the discounted payoff of a strangle. In the case when $r_e \leq 0\%$, all prices decrease as r_e increases. However, in the case when $r_e \geq 0$, all prices increase as r_e increases.

Example 5.49. We know from Theorem 2.14 and (2.26)-(2.27) that the superhedging prices are independent of market probabilities. However, the indifference prices may depend on it. In Figure 5.10, we present $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;0)$, $\widetilde{\pi}_{\mathrm{F}}^{bi}(c;0)$, and $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;0) - \widetilde{\pi}_{\mathrm{F}}^{bi}(c;0)$ for various p, where $r_e = 0\%$, $k = 0.5\%$, $\alpha_t = 1$ for all $t \in \mathcal{I}$, and $\bar{c}_t = 0$ for all $t = 0, \dots, T$. We set $\sum_{t=0}^{T} c_t$ to be $\mathrm{Call}_T^{\mathrm{C}}(100)$, $\mathrm{Put}_T^{\mathrm{C}}(100)$, $\mathrm{But}_T^{\mathrm{C}}(90, 110)$ respectively in Figures 5.10(a)-5.10(c). It shows that indifference prices can be affected by p. Moreover, in all three subfigures, the difference between seller's and buyer's indifference prices $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;0) - \widetilde{\pi}_{\mathrm{F}}^{bi}(c;0)$ is increasing in p when $p \leq 0.5$, and it is decreasing in p when $p \geq 0.5$, so it reaches its maximum at $p = 0.5$.

Example 5.50. In our setting, we have $S_0^b = S_0^a$. By Corollary 5.9, the indifference prices satisfies (5.15) (i.e. remain unchanged for two endowments $(\bar{c}_t)_{t=0}^{T}$ and $(\bar{c}'_t)_{t=0}^{T}$) if $\sum_{t=0}^{T} \bar{c}_t - \sum_{t=0}^{T} \bar{c}'_t$ is a constant. This example shows that

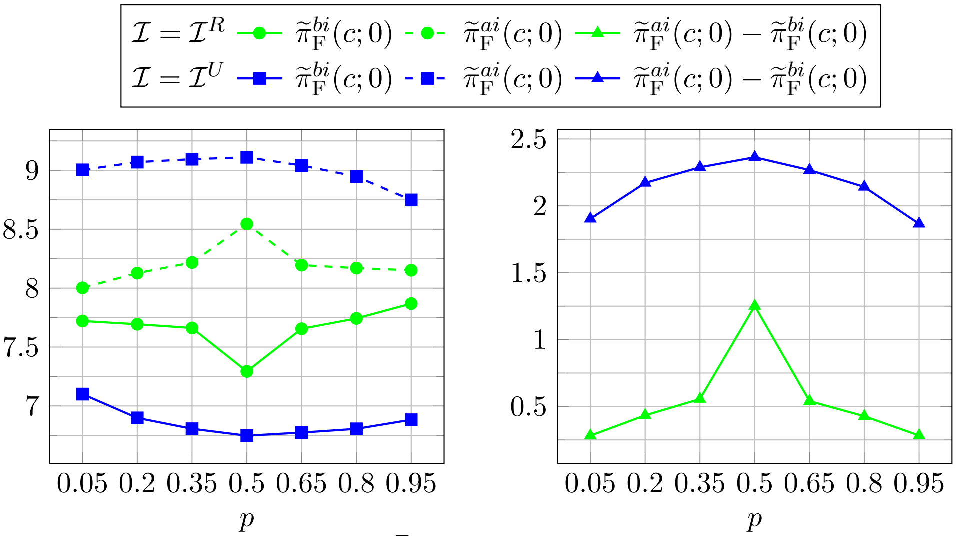


(a) $\sum_{t=0}^{T} c_t = \frac{1}{1+r_e}\mathrm{Call}_T^{\mathrm{C}}(100)$

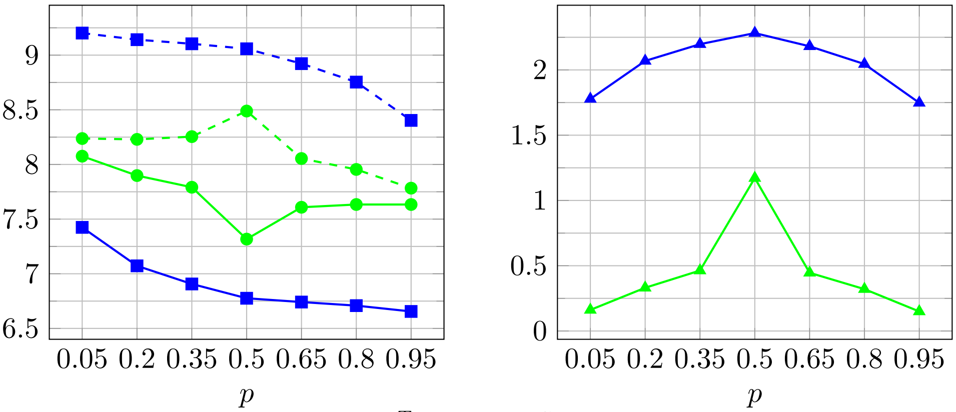
(b) $\sum_{t=0}^{T} c_t = \frac{1}{1+r_e}\mathrm{Put}_T^{\mathrm{C}}(100)$

(c) $\sum_{t=0}^{T} c_t = \frac{1}{1+r_e}\mathrm{Str}_T^{\mathrm{C}}(95, 105)$

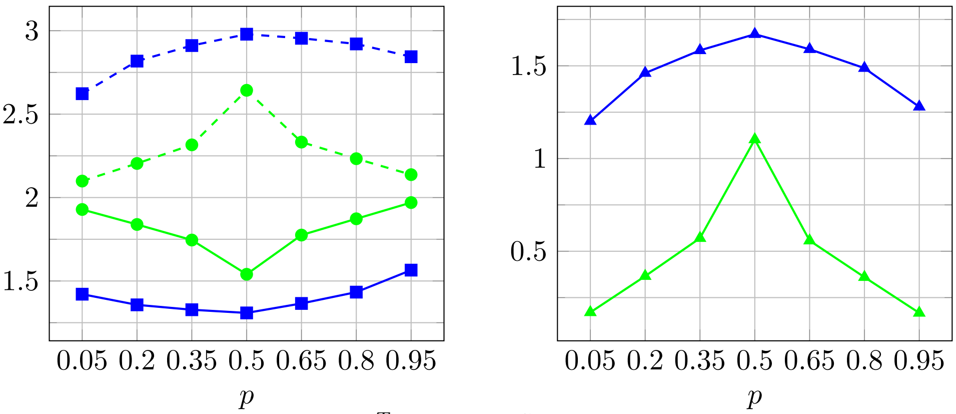
Figure 5.9: Indifference prices $\widetilde{\pi}_{\mathrm{F}}^{bi}(c;0)$ and $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;0)$ for various values of r_e, where $T = 52$, $p = 0.5$, $k = 0.5\%$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$



(a) $\sum_{t=0}^{T} c_t = \mathrm{Call}^{\mathrm{C}}_T(100)$

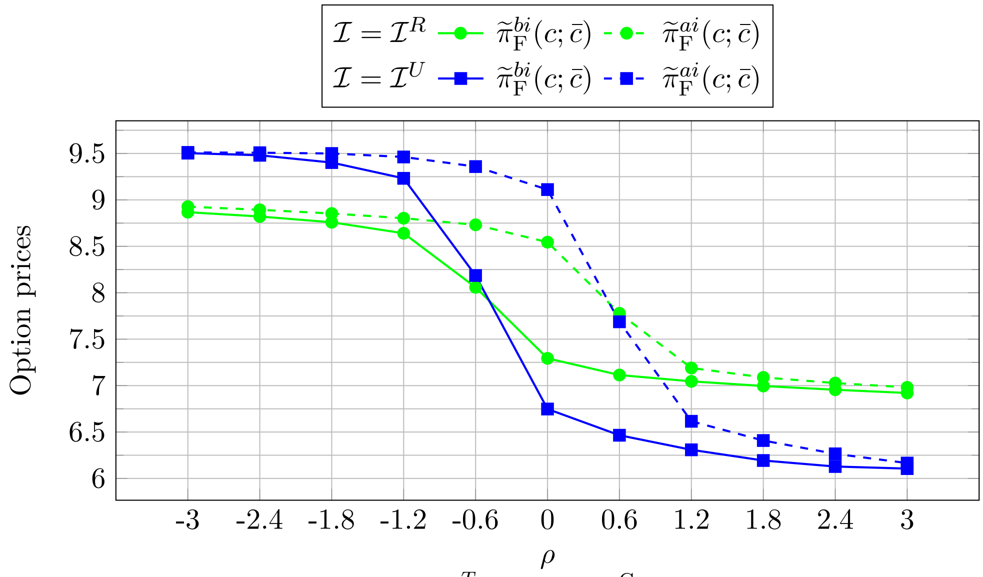


(b) $\sum_{t=0}^{T} c_t = \mathrm{Put}^{\mathrm{C}}_T(100)$

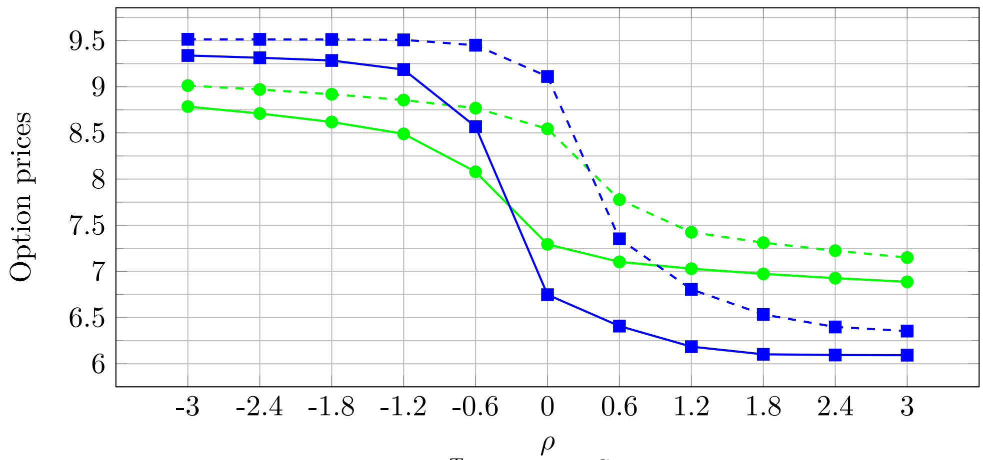


(c) $\sum_{t=0}^{T} c_t = \mathrm{But}^{\mathrm{C}}_T(90, 110)$

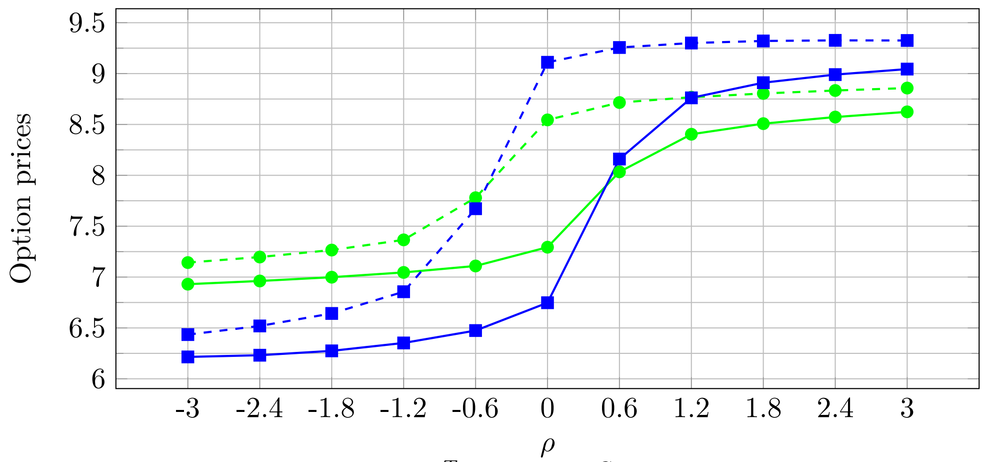
Figure 5.10: Indifference prices $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;0)$ and $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;0)$ for various values of p, where $T = 52$, $r_e = 0\%$, $k = 0.5\%$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$



(a) $\sum_{t=0}^{T} \bar{c}_t = \rho \mathrm{Call}^{\mathrm{C}}_T(100)$



(b) $\sum_{t=0}^{T} \bar{c}_t = \rho \mathrm{Str}^{\mathrm{C}}_T(90, 110)$



(c) $\sum_{t=0}^{T} \bar{c}_t = \rho \mathrm{But}^{\mathrm{C}}_T(80, 120)$

Figure 5.11: Indifference prices $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;\bar{c})$ and $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;\bar{c})$ for various endowment $\bar{c}$, where $\sum_{t=0}^{T} c_t = \mathrm{Call}^{\mathrm{C}}_T(100)$, $T = 52$, $p = 0.5$, $r_e = 0\%$, $k = 0.5\%$, and $\alpha_t = \frac{1}{2}$ for all $t \in \mathcal{I}$

(5.15) may not hold true if $\sum_{t=0}^{T} \bar{c}_t - \sum_{t=0}^{T} \bar{c}'_t$ is not a constant. Let $p = 0.5$, $r_e = 0\%$, $k = 0.5\%$, and $\alpha_t = \frac{1}{2}$ for all $t \in \mathcal{I}$. In Figure 5.11, we consider the indifference prices of $c = (c_t)_{t=0}^{T}$ such that $\sum_{t=0}^{T} c_t = \mathrm{Call}_T^{\mathrm{C}}(100)$ for various different $\bar{c} = (\bar{c}_t)_{t=0}^{T}$. Firstly, in Figures 5.11(a)-5.11(c), the endowment $\bar{c}$ is set to satisfy

$$\sum_{t=0}^{T} \bar{c}_t = \rho \mathrm{Call}_T^{\mathrm{C}}(100)$$

$$\sum_{t=0}^{T} \bar{c}_t = \rho \mathrm{Str}_T^{\mathrm{C}}(90, 110)$$

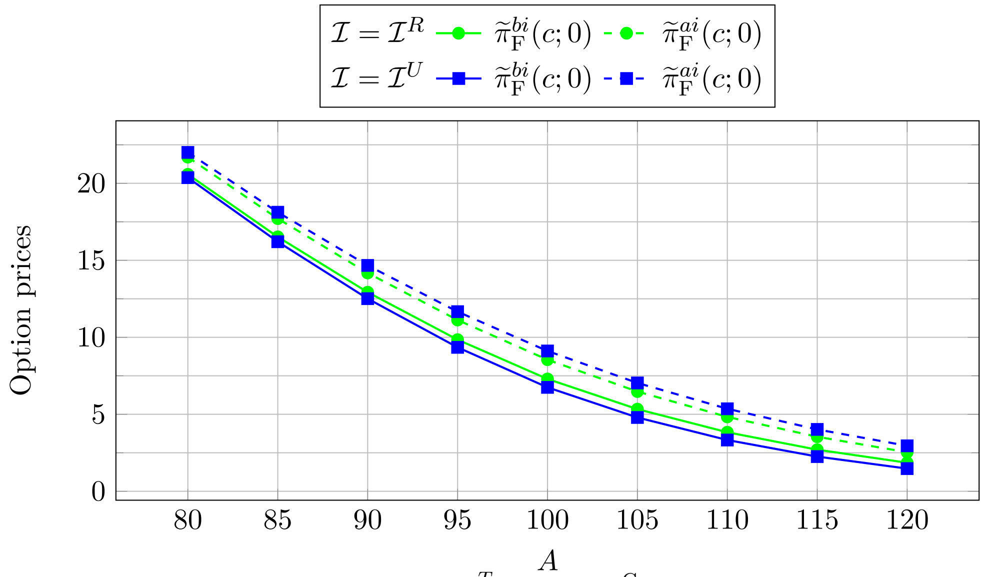
$$\sum_{t=0}^{T} \bar{c}_t = \rho \mathrm{But}_T^{\mathrm{C}}(80, 120)$$

respectively, where $-3 \leq \rho \leq 3$ is a scalar. It shows that, in both Figures 5.11(a) and 5.11(b), the indifference prices decrease as ρ increases. However, in Figure 5.11(c), as ρ increases, the indifference prices increase. In all three subfigure, as ρ increases from 0 to 3 or decreases from 0 to -3, the difference between seller's and buyer's indifference prices becomes smaller. The conclusion is that investor's endowment $\bar{c}$ can affect the indifference prices $\widetilde{\pi}_{\mathrm{F}}^{ai}(c; \bar{c})$ and $\widetilde{\pi}_{\mathrm{F}}^{bi}(c; \bar{c})$, but more endowment may not produce a lower/higher indifference price.

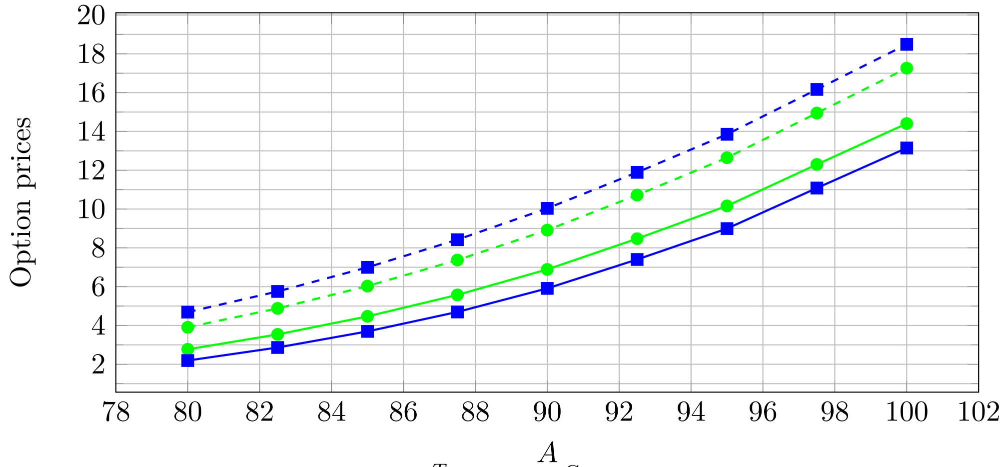
Example 5.51. Let $p = 0.5$, $r_e = 0\%$, $k = 0.5\%$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$, and $\bar{c}_t = 0$ for all $t = 0, \ldots, T$. We are going to compute the indifference prices of c for various values of strike prices. Firstly, we consider $\sum_{t=0}^{T} c_t = \mathrm{Call}_T^{\mathrm{C}}(A)$ in Figure 5.12(a) for various values of strike price A. It shows that all indifference prices decrease as A increases. As expected, a higher value of option payoff leads to a higher price. Similar pattern can be observed when $\sum_{t=0}^{T} c_t$ is the payoff of a strangle or a butterfly; see Figures 5.12(b) and 5.12(c).

Example 5.52. In this example, we set $p = 0.5$, $r_e = 0\%$, $\alpha_t = 1$ for all $t \in \mathcal{I}$, and $\bar{c}_t = 0$ for all $t = 0, \ldots, T$. In Figure 5.13, we present the indifference prices of c for various values of k which is the transaction costs parameter. First of all, in Figure 5.13(a), we take $\sum_{t=0}^{T} c_t$ as the payoff of a call option delivered by portfolio. Moreover, we set $\sum_{t=0}^{T} c_t$ as the payoff of a strangle and a butterfly respectively in Figures 5.13(b) and 5.13(c). It suggests that, when the value of k increases, the gap between seller's and buyer's indifference prices becomes bigger. In the case when $k = 0\%$ (i.e. there is no transaction costs), all indifference prices are the same.

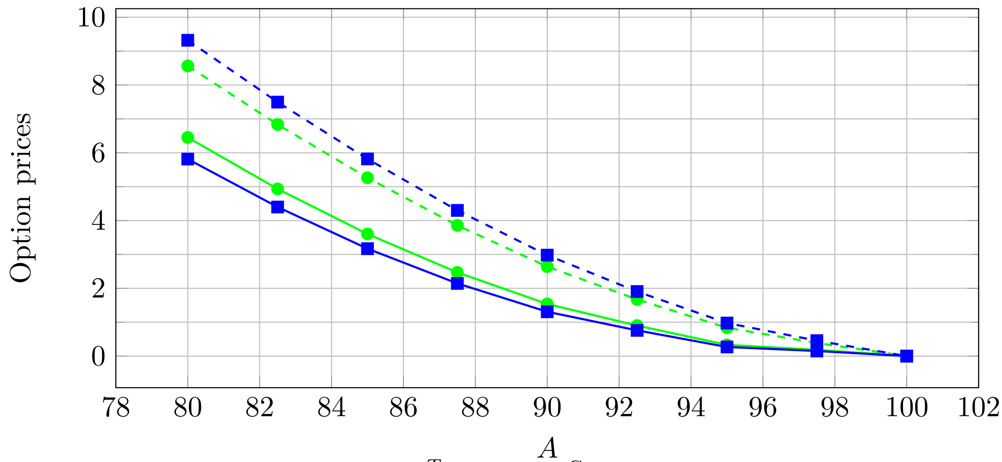
Example 5.53. First of all, let $p = 0.5$, $r_e = 0\%$, $k = 0.5\%$, and $\bar{c}_t = 0$ for all $t = 0, \ldots, T$. Moreover, the process $(\alpha_t)_{t=0}^{T}$ is set to be $\alpha_t = \alpha$ for



(a) $\sum_{t=0}^{T} c_t = \mathrm{Call}_T^{\mathrm{C}}(A)$

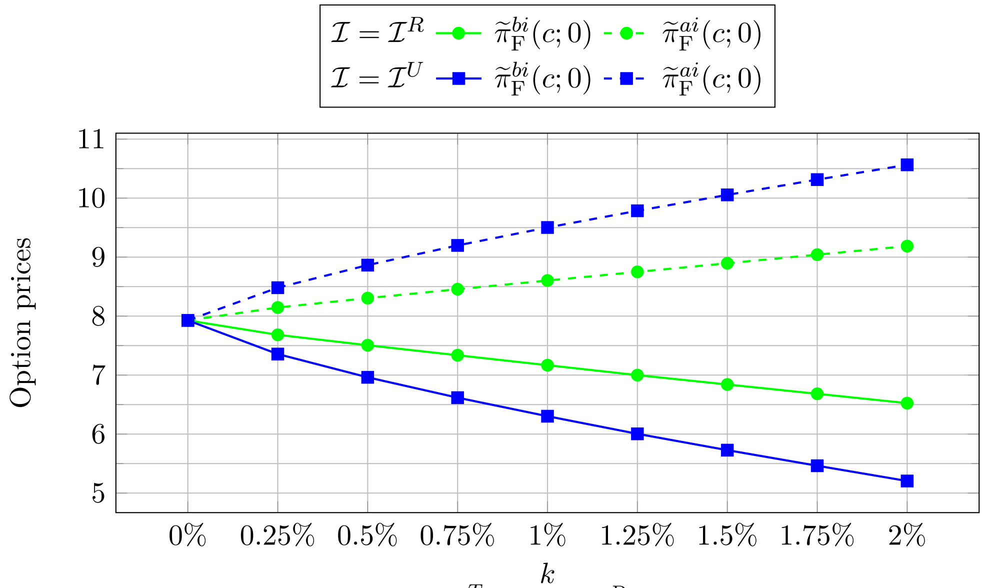


(b) $\sum_{t=0}^{T} c_t = \mathrm{Str}_T^{\mathrm{C}}(A, 200 - A)$

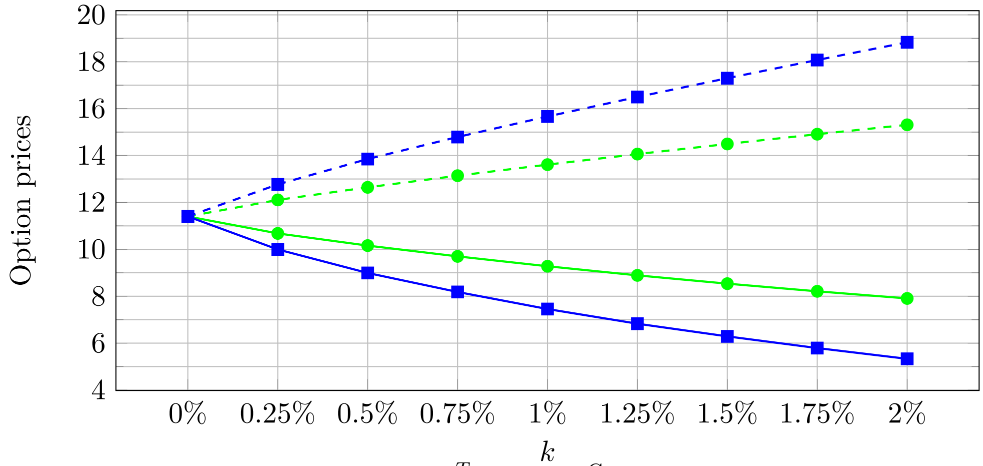


(c) $\sum_{t=0}^{T} c_t = \mathrm{But}_T^{\mathrm{C}}(A, 200 - A)$

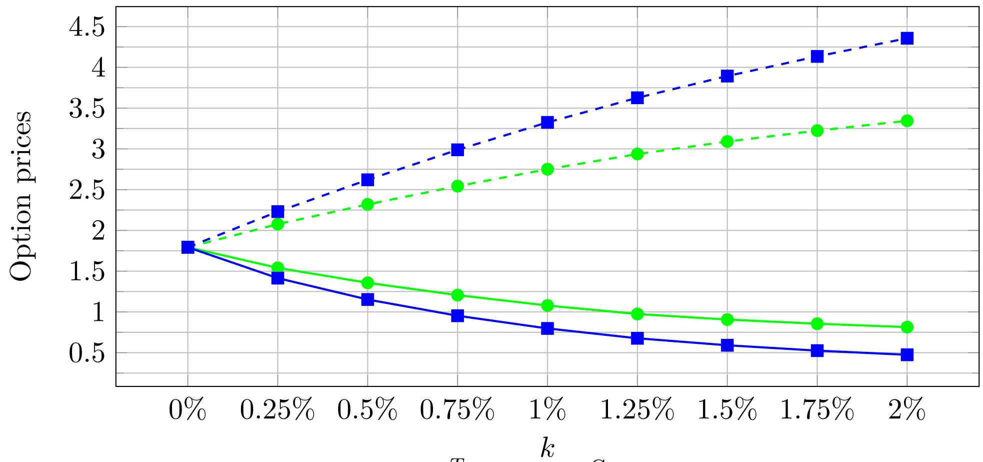
Figure 5.12: Indifference prices $\widetilde{\pi}_{\mathrm{F}}^{bi}(c;0)$ and $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;0)$ for various strike prices, where $T = 52$, $p = 0.5$, $r_e = 0\%$, $k = 0.5\%$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$



(a) $\sum_{t=0}^{T} c_t = \mathrm{Call}^{\mathrm{P}}_T(100)$

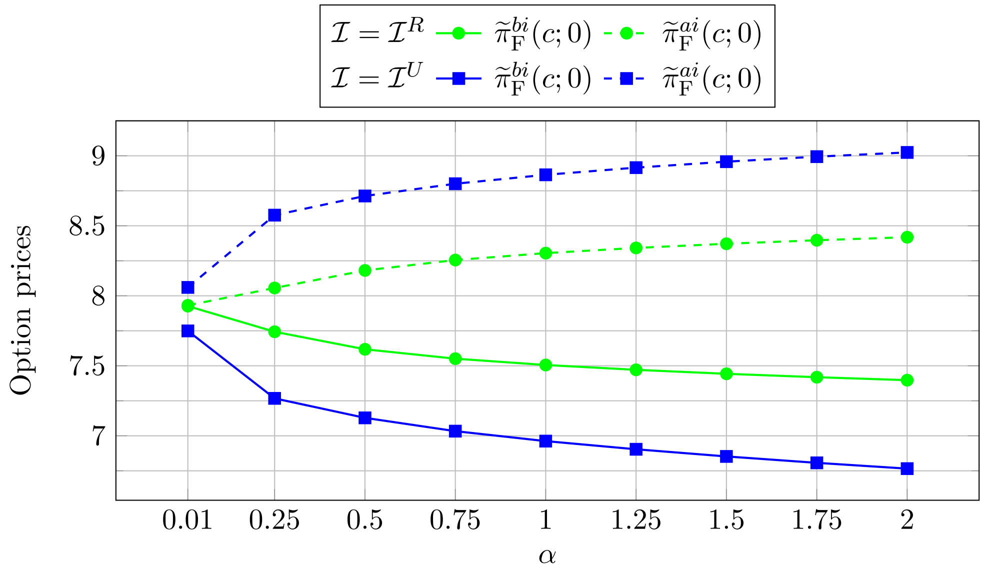


(b) $\sum_{t=0}^{T} c_t = \mathrm{Str}^{\mathrm{C}}_T(95, 105)$

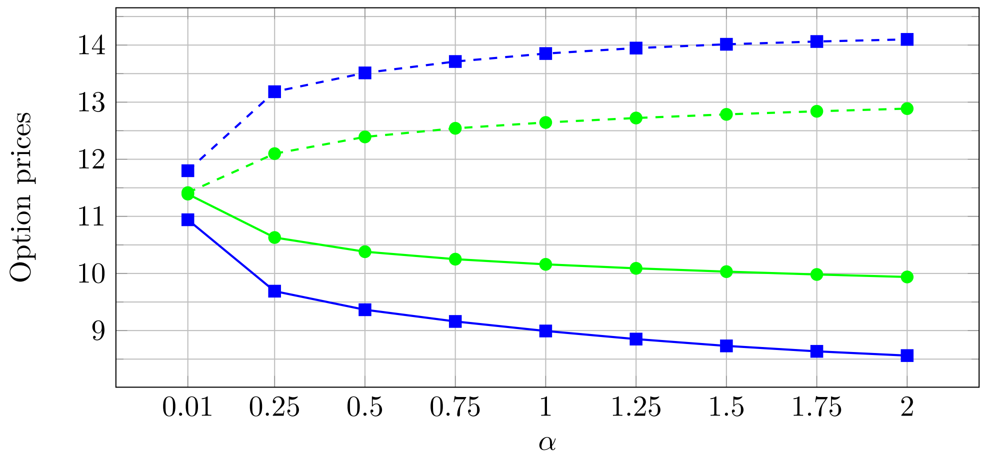


(c) $\sum_{t=0}^{T} c_t = \mathrm{But}^{\mathrm{C}}_T(95, 115)$

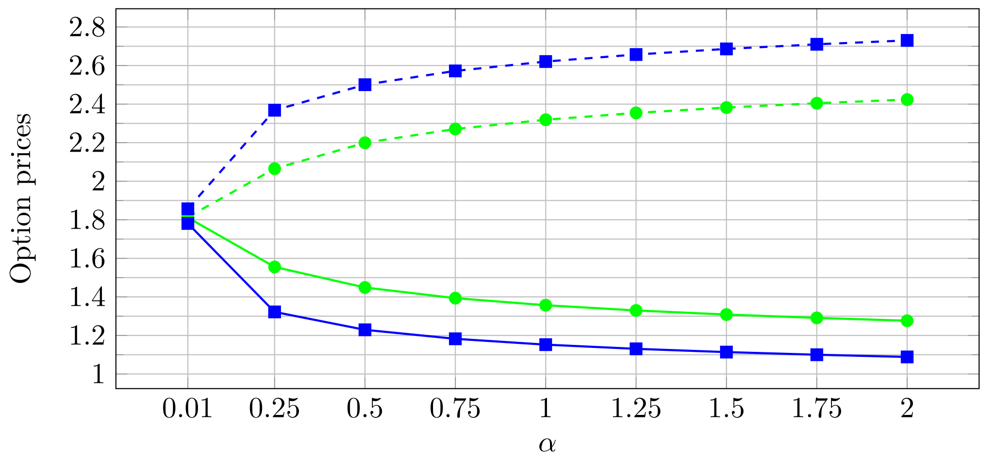
Figure 5.13: Indifference prices $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;0)$ and $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;0)$ for various values of k, where $T = 52$, $p = 0.5$, $r_e = 0\%$, and $\alpha_t = 1$ for all $t \in \mathcal{I}$



(a) $\sum_{t=0}^{T} c_t = \mathrm{Call}^{\mathrm{P}}_T(100)$



(b) $\sum_{t=0}^{T} c_t = \mathrm{Str}^{\mathrm{C}}_T(95, 105)$



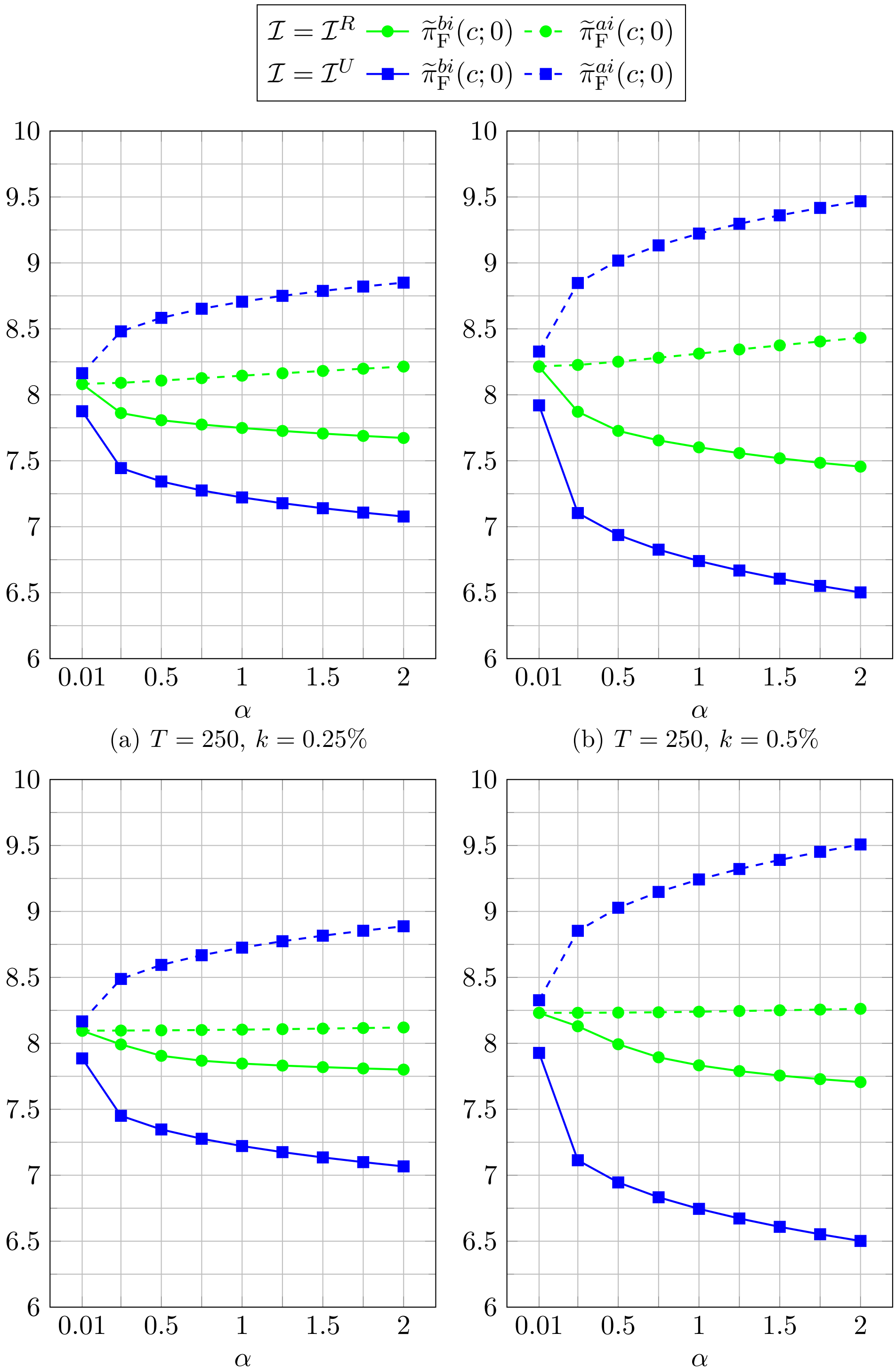
(c) $\sum_{t=0}^{T} c_t = \mathrm{But}^{\mathrm{C}}_T(95, 115)$

Figure 5.14: Indifference prices $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;0)$ and $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;0)$ for various α, where $T = 52$, $p = 0.5$, $r_e = 0\%$, $k = 0.5\%$, $\alpha_t = \alpha$ for all $t \in \mathcal{I}$

all $t \in \mathcal{I}$, so α is the risk aversion coefficient at each time step $t \in \mathcal{I}$. In Figure 5.14, we consider the indifference prices of c for 9 different values of α ranging from 0.01 to 2. The value $\sum_{t=0}^{T} c_t$ is set to be the payoff of a call, a strangle, and a butterfly respectively in Figures 5.14(a)-5.14(c). In all cases, as α increases, the buyer's indifference price decreases, and the seller's indifference price increases. This suggests that if the seller and the buyer are willing to take more risks which corresponds to a lower value of α, then the gap between seller's and buyer's indifference prices will be smaller.

This section ends with the following example which presents the indifference prices and superhedging prices in the market models with larger number of steps T. It shows that the bid-ask indifference price interval can be much narrower than the no-arbitrage price interval.

Example 5.54. In our final example, let $p = 0.5$, $r_e = 0\%$, and $\bar{c}_t = 0$ for all $t = 0, \ldots, T$. Similar to Example 5.53, we take $\alpha_t = \alpha$ for all $t \in \mathcal{I}$. In Figures 5.15-5.18, we present the indifference prices of a call, a put, a strangle, and a butterfly respectively in the market models with $T = 250, 1000$ and $k = 0.25\%, 0.5\%$. In addition, in every figure, the prices are computed for 9 different values of α ranging from 0.01 to 2. The superhedging prices for these four options are presented in Table 5.11, and all the prices in this table are provided by Dr. Alet Roux by using the method from Roux & Zastawniak (2016). The difference between seller's and buyer's indifference prices appears to be much smaller than the difference between seller's and buyer's superhedging prices. This is especially the case when $k = 0.5\%$ and $T = 1000$.



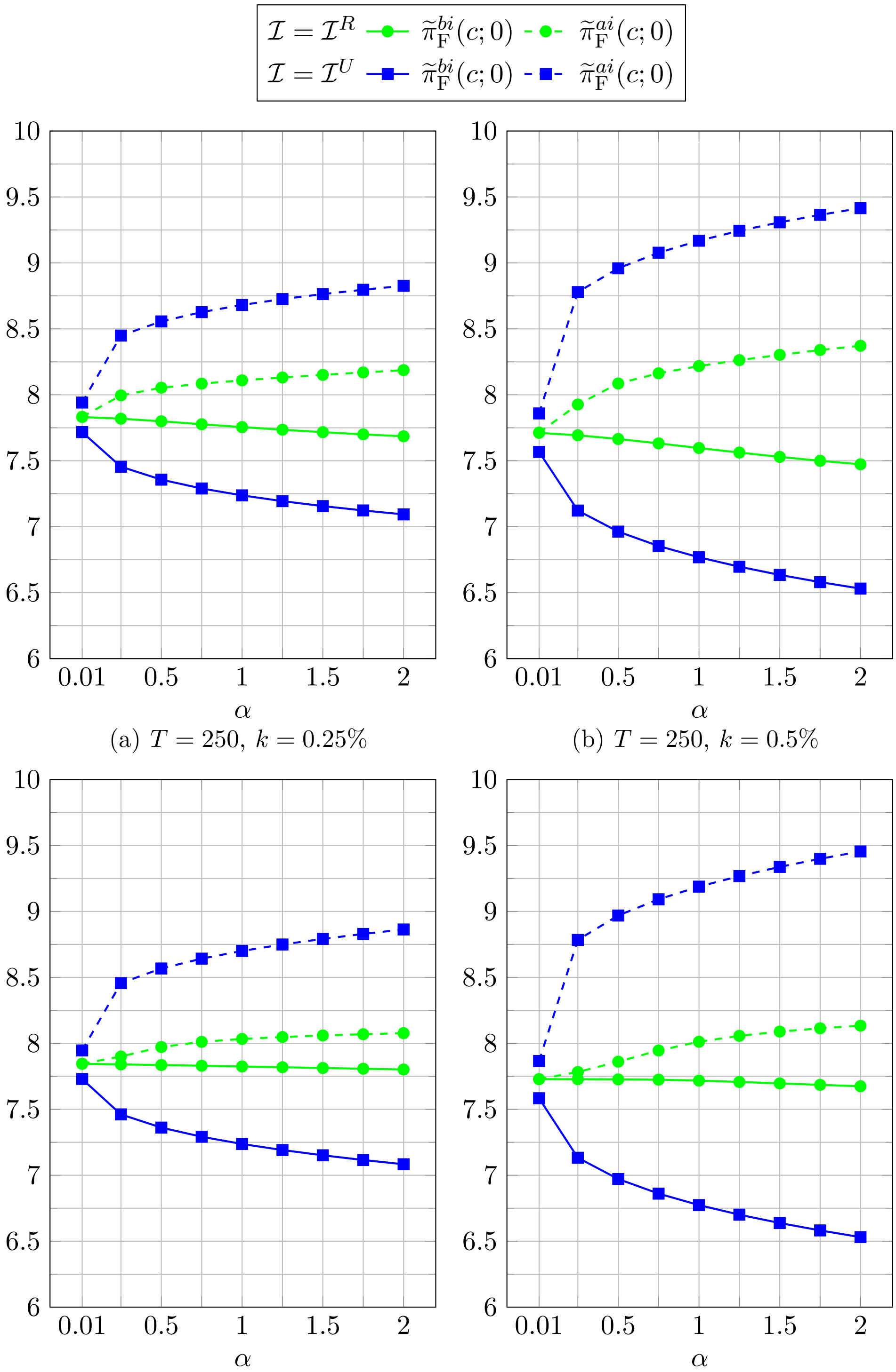
(a) $T = 250,\ k = 0.25\%$

(b) $T = 250,\ k = 0.5\%$

(c) $T = 1000,\ k = 0.25\%$

(d) $T = 1000,\ k = 0.5\%$

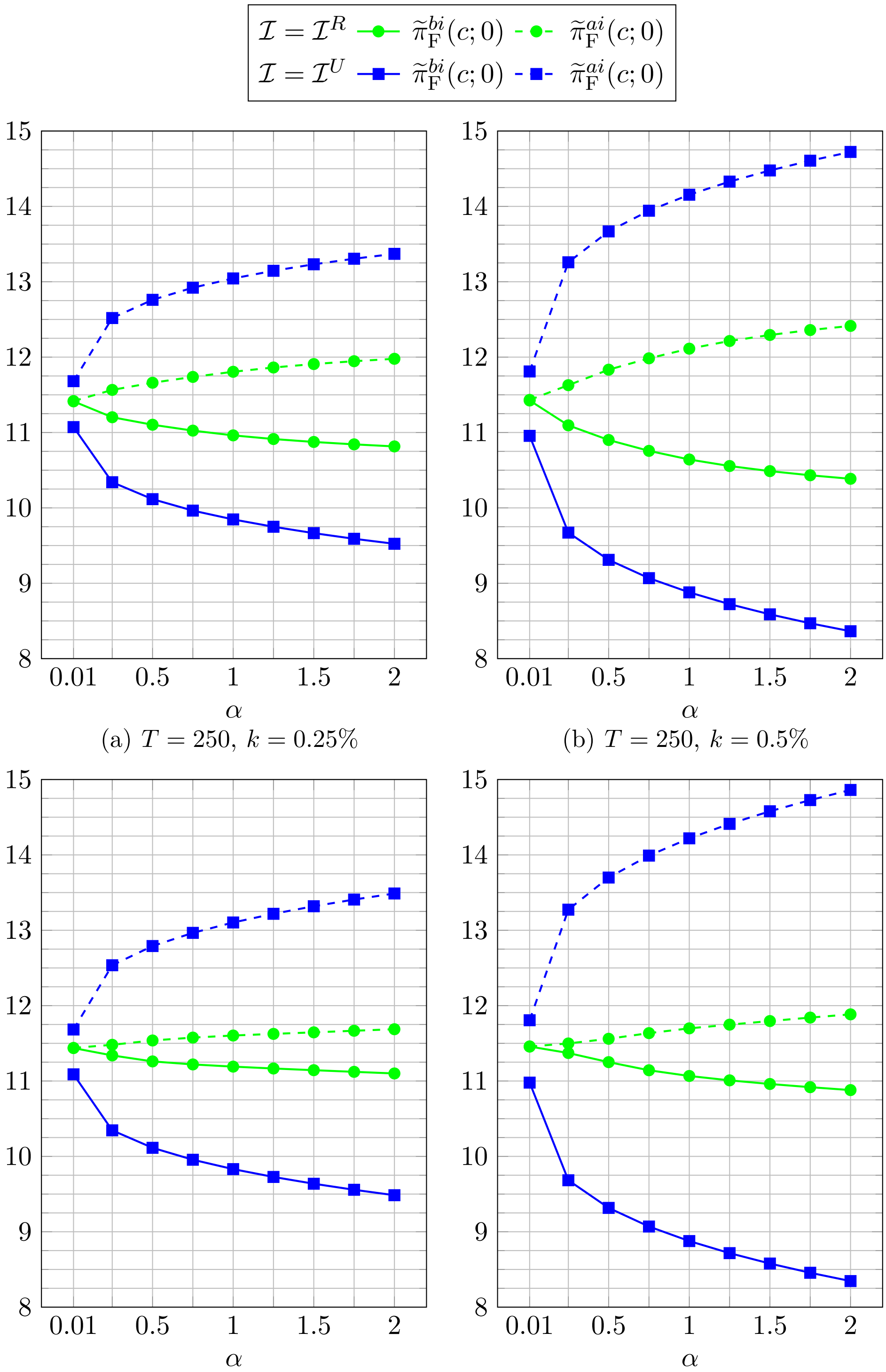
Figure 5.15: Indifference prices $\widetilde{\pi}_{\mathrm{F}}^{bi}(c;0)$ and $\widetilde{\pi}_{\mathrm{F}}^{ai}(c;0)$ for various α, where $\sum_{t=0}^{T} c_t = \mathrm{Call}_T^{\mathrm{C}}(100)$, $p = 0.5$, $r_e = 0\%$, and $\alpha_t = \alpha$ for all $t \in \mathcal{I}$



(a) $T = 250,\ k = 0.25\%$ (b) $T = 250,\ k = 0.5\%$

(c) $T = 1000,\ k = 0.25\%$ (d) $T = 1000,\ k = 0.5\%$

Figure 5.16: Indifference prices $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;0)$ and $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;0)$ for various α, where $\sum_{t=0}^{T} c_t = \mathrm{Put}^{\mathrm{C}}_T(100)$, $p = 0.5$, $r_e = 0\%$, and $\alpha_t = \alpha$ for all $t \in \mathcal{I}$



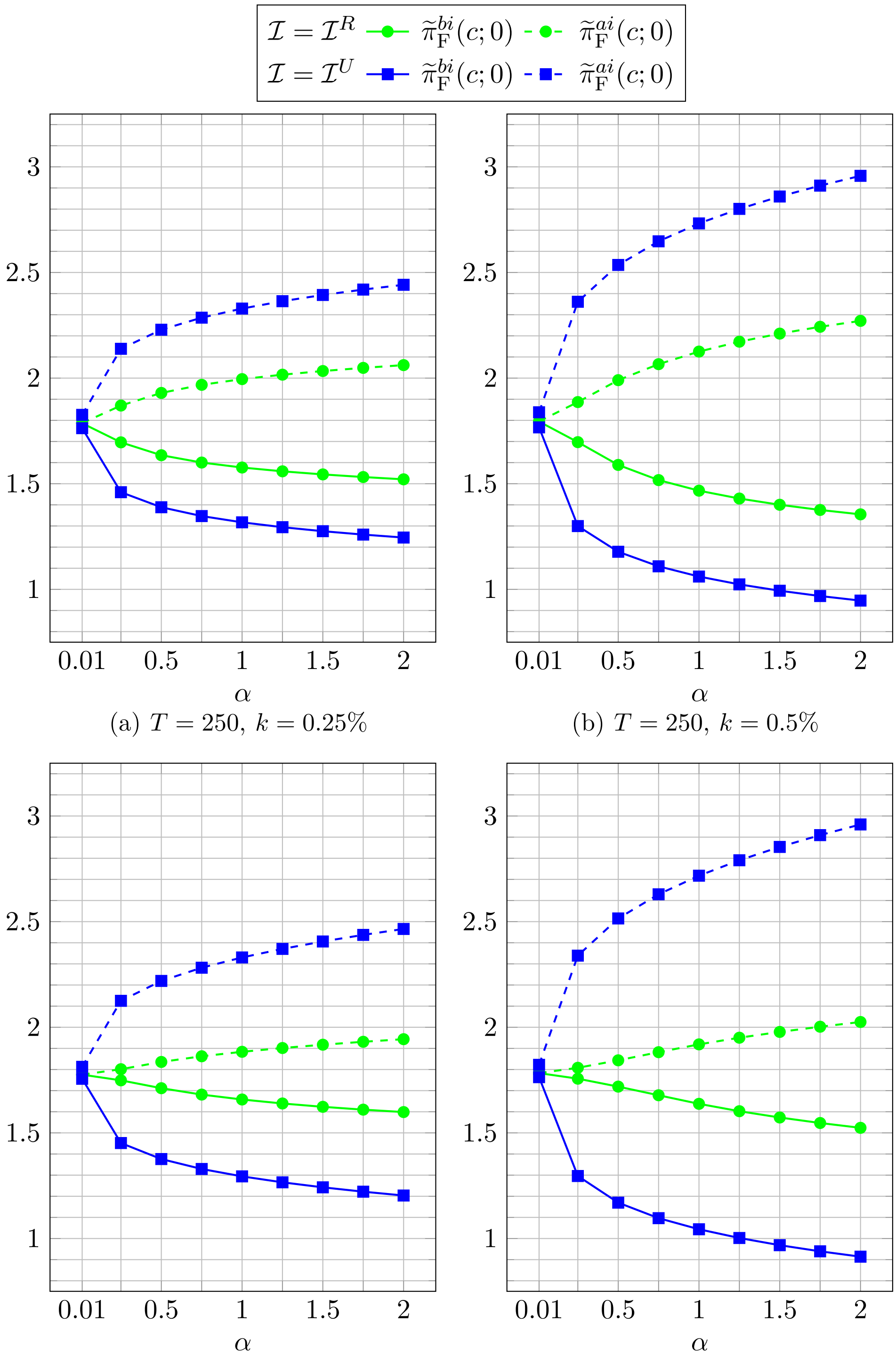
(a) $T = 250$, $k = 0.25\%$

(b) $T = 250$, $k = 0.5\%$

(c) $T = 1000$, $k = 0.25\%$

(d) $T = 1000$, $k = 0.5\%$

Figure 5.17: Indifference prices $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;0)$ and $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;0)$ for various α, where $\sum_{t=0}^{T} c_t = \mathrm{Str}^{\mathrm{C}}_T(95, 105)$, $p = 0.5$, $r_e = 0\%$, and $\alpha_t = \alpha$ for all $t \in \mathcal{I}$



(a) $T = 250,\ k = 0.25\%$

(b) $T = 250,\ k = 0.5\%$

(c) $T = 1000,\ k = 0.25\%$

(d) $T = 1000,\ k = 0.5\%$

Figure 5.18: Indifference prices $\widetilde{\pi}^{bi}_{\mathrm{F}}(c;0)$ and $\widetilde{\pi}^{ai}_{\mathrm{F}}(c;0)$ for various α, where $\sum_{t=0}^{T} c_t = \mathrm{But}^{\mathrm{C}}_T(95, 115)$, $p = 0.5$, $r_e = 0\%$ and $\alpha_t = \alpha$ for all $t \in \mathcal{I}$

T	$k=0.25\%$ buyer	$k=0.25\%$ seller	$k=0.5\%$ buyer	$k=0.5\%$ seller
	$\sum_{t=0}^{T} c_t = \mathrm{Call}_T^{\mathrm{C}}(100)$			
250	6.06057	9.53113	3.39402	10.91410
1000	3.52132	10.78134	0.00000	13.04456
	$\sum_{t=0}^{T} c_t = \mathrm{Put}_T^{\mathrm{C}}(100)$			
250	6.07543	9.50702	3.40983	10.85840
1000	3.52983	10.75411	0.00000	12.97824
	$\sum_{t=0}^{T} c_t = \mathrm{Str}_T^{\mathrm{C}}(95, 105)$			
250	7.84118	14.41934	3.08667	17.07163
1000	3.22336	16.86926	0.00000	21.26935
	$\sum_{t=0}^{T} c_t = \mathrm{But}_T^{\mathrm{C}}(95, 115)$			
250	0.88800	3.06615	0.23926	5.29936
1000	0.25824	5.14877	0.00002	8.51486

Table 5.11: Buyer's superhedging price $\pi_{\mathrm{F}}^{b}(c) = \pi_{\mathrm{E}}^{b}(\sum_{t=0}^{T} c_t)$ and seller's superhedging price $\pi_{\mathrm{F}}^{a}(c) = \pi_{\mathrm{E}}^{a}(\sum_{t=0}^{T} c_t)$ for various $\sum_{t=0}^{T} c_t$, k, and T, where $r_e = 0\%$

Appendix A

Mathematical background

A.1 Mathematical preliminaries

Let X be a vector space and f be an $\mathbb{R} \cup \{\infty\}$-valued function on a convex set $S \subseteq X$. We call f a *convex function* if for any $x, y \in S$ and $\lambda \in (0,1)$ we have

$$f(\lambda x + (1-\lambda) y) \leq \lambda f(x) + (1-\lambda) f(y).$$

Moreover, a function f is called *proper* if $\operatorname{dom} f \neq \emptyset$ where

$$\operatorname{dom} f := \{x \in S \mid f(x) < \infty\}$$

is the *effective domain* of f. The *epigraph* of f is defined as

$$\operatorname{epi} f := \{(x, y) \mid x \in S,\ y \in \mathbb{R},\ y \geq f(x)\} \subseteq X \times \mathbb{R}.$$

Notice that f is a convex function if and only if $\operatorname{epi} f$ is convex.

The function f is called *lower semicontinuous* (Rockafellar 1974, p. 14) if the set $\operatorname{epi} f$ is closed in $X \times \mathbb{R}$. The notion of lower semicontinuity depends on $X \times \mathbb{R}$ being a topological space, and this thesis only considers the cases when $X = \mathbb{R}$ and $X = \mathcal{N}^2$. The following remark discusses the notion of closedness in $\mathcal{N}^2 \times \mathbb{R}$ used in this thesis.

Remark A.1. Recall that Ω is finite. Let $|\Omega|$ be the number of elements in Ω. For any $x \in \mathcal{L}_T^2$, the value of x can be represented by the vector $R(x)$ as

$$R(x) := \left(x^1(\omega_1), x^2(\omega_1), \ldots, x^1(\omega_{|\Omega|}), x^2(\omega_{|\Omega|})\right) \in \mathbb{R}^{2|\Omega|}.$$

Similarly, for any $(x, y) = ((x_t)_{t=0}^T, y) \in \mathcal{N}^2 \times \mathbb{R}$ we have $x_t \in \mathcal{L}_t^2 \subseteq \mathcal{L}_T^2$ for all

$t = 0, \ldots, T$, and the value of (x, y) can be represented by

$$P((x,y)) := (R(x_0), \ldots, R(x_T), y) \in \mathbb{R}^{2|\Omega|(T+1)+1}.$$

We call a set A in $\mathcal{L}_T^2$ closed if the set $\{R(x) \mid x \in A\}$ is closed in $\mathbb{R}^{2|\Omega|}$. In addition, we call a set A in $\mathcal{N}^2 \times \mathbb{R}$ closed if the set $\{P(x) \mid x \in A\}$ is closed in $\mathbb{R}^{2|\Omega|(T+1)+1}$.

An $\mathbb{R} \cup \{\pm\infty\}$-valued function on $B \subseteq \mathbb{R}$ is said to be continuous on $B' \subseteq B$ if the restriction of this function to B' is a continuous function.

Lemma A.2. *If $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is proper, convex, and lower semicontinuous, then f is continuous on* $\operatorname{dom} f$.

Proof. From Theorem 10.1 of Rockafellar (1997), the function f is continuous on $\operatorname{ri}\operatorname{dom} f$, where $\operatorname{ri} A$ is the relative interior of a given set A. Moreover, the function f is continuous on any closed subinterval of $\operatorname{dom} f$ (Rockafellar 1997, Theorems 10.2 and 20.5). Notice that $\operatorname{dom} f$ is non-empty and convex because f is proper and convex. Then $\operatorname{dom} f$ is an interval, in other words, it is the union of $\operatorname{ri}\operatorname{dom} f$ and zero, one or two endpoints. For convenience, we define

$$\begin{aligned} a &:= \inf \operatorname{dom} f, \\ b &:= \sup \operatorname{dom} f. \end{aligned}$$

We are going to show that f is continuous on $\operatorname{dom} f$ by considering the following three situations for $\operatorname{dom} f$.

1. If $\operatorname{dom} f = (a, b)$ (i.e. $\operatorname{dom} f$ contains no endpoint), then f is continuous on $\operatorname{dom} f$ because $\operatorname{dom} f = \operatorname{ri}\operatorname{dom} f$.

2. If $\operatorname{dom} f = [a, b]$ (i.e. $\operatorname{dom} f$ contains both endpoints), then $\operatorname{dom} f$ is a closed subinterval of $\operatorname{dom} f$. This means that f is continuous on $\operatorname{dom} f$.

3. In the case when $\operatorname{dom} f = [a, b)$ or $\operatorname{dom} f = (a, b]$ (i.e. $\operatorname{dom} f$ contains exactly one endpoint), the function f is continuous on (a, b) because $\operatorname{ri}\operatorname{dom} f = (a, b)$. Let

   $$\epsilon := \min\left\{\frac{b-a}{2}, 1\right\} \in (0, 1].$$

   If $\operatorname{dom} f = [a, b)$, the function f is continuous on $[a, a+\epsilon]$ because $[a, a+\epsilon]$ is a closed subinterval of $\operatorname{dom} f$. Since the intersection

   $$[a, a+\epsilon] \cap (a, b) = (a, a+\epsilon)$$

contains infinitely many points, the function f is continuous on

$$[a, a+\epsilon] \cup (a, b) = \operatorname{dom} f.$$

Similarly, if $\operatorname{dom} f = (a, b]$, then f is continuous on $[b-\epsilon, b]$ because $[b-\epsilon, b]$ is a closed subinterval of $\operatorname{dom} f$. Since the intersection

$$(a, b) \cap [b-\epsilon, b] = (b-\epsilon, b)$$

contains infinitely many points, the function f is continuous on

$$(a, b) \cup [b-\epsilon, b] = \operatorname{dom} f.$$

This completes the proof. $\square$

The *recession cone* of a nonempty set $C \subseteq X$ is defined as

$$C^{\infty} := \{y \in X \mid x + \lambda y \in C \text{ for all } x \in C \text{ and } \lambda \geq 0\}.$$

Observe that 0 is contained in any recession cone. The following result says that if a convex cone contains 0, then it is equal to its recession cone.

Lemma A.3. *Let $C \subseteq X$ be a convex cone such that $0 \in C$. Then $C = C^{\infty}$.*

Proof. Suppose that $y \in C$. For all $x \in C$ and $\lambda \geq 0$, we have $\lambda y \in C$ because C is a cone that contains 0. Then

$$x + \lambda y = 2\left(\frac{1}{2}x + \frac{1}{2}\lambda y\right) \in C$$

because C is convex cone. Thus $y \in C^{\infty}$, and hence $C \subseteq C^{\infty}$. The opposite inclusion also holds true. Suppose that $y \in C^{\infty}$. Let $\lambda > 0$. It follows from $0 \in C$ and the definition of C^{∞} that $\lambda y = 0 + \lambda y \in C$. Then $y \in C$ because C is a cone. Thus $C^{\infty} \subseteq C$. The result follows. $\square$

The *recession function* of a proper convex function f is defined as the function f^{∞} such that

$$\operatorname{epi} f^{\infty} = (\operatorname{epi} f)^{\infty}.$$

Example A.4. In this example, we will compute the recession functions of a number of $\mathbb{R} \cup \{\infty\}$-valued convex functions on $\mathbb{R}$.

1. Let $a \in \mathbb{R}$, and let f be a linear function of the form $f(x) = ax$ for all $x \in \mathbb{R}$. Then

$$\operatorname{epi} f = \{(x, y) \mid y \geq ax\}$$

is a half space and hence a convex cone containing 0. It follows from Lemma A.3 that

$$\operatorname{epi} f = (\operatorname{epi} f)^{\infty}.$$

Thus $f^{\infty} = f$.

2. Let $\alpha > 0$ and $v(x) = e^{\alpha x} - 1$ for all $x \in \mathbb{R}$. Then v is an exponential regret function defined in Example 3.4.1. Fix any $(x, y) \in \mathbb{R}^2$, and consider the following three situations. If $y < 0$, then $(x, y) \notin (\operatorname{epi} v)^{\infty}$ because v is bounded from below. Moreover, if $x > 0$ and $y \geq 0$, then $(x, y) \notin (\operatorname{epi} v)^{\infty}$ because v and v' are always increasing. However, if $x \leq 0$ and $y \geq 0$, then $(x, y) \in (\operatorname{epi} v)^{\infty}$. Thus

$$(\operatorname{epi} v)^{\infty} = \{(x, y) |\, x \leq 0, y \geq 0\} = (-\infty, 0] \times [0, \infty).$$

To ensure $\operatorname{epi} v^{\infty} = (\operatorname{epi} v)^{\infty}$, we must have $v^{\infty} = \delta_{(-\infty,0]}$, where $\delta_{(-\infty,0]}$ is the indicator function defined in (3.1).

3. Let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a proper convex function with a bounded effective domain. Then

$$(\operatorname{epi} f)^{\infty} = \{(0, y) |\, y \geq 0\},$$

which implies that $f^{\infty} = \delta_{\{0\}}$.

This section ends with three technical results in Lemmas A.5-A.7.

Lemma A.5. *Let f be a closed proper convex function on $\mathbb{R}$. Suppose that $(x, y) \notin \operatorname{epi} f^{\infty}$ for some $x < 0$ and $y > 0$ and that $(x', y') \notin \operatorname{epi} f^{\infty}$ for some $x' > 0$ and $y' > 0$. Then the function f attains its infimum.*

Proof. It is sufficient to show that f is neither a nondecreasing function nor a nonincreasing function. Then f attains its infimum; see Theorem 27.2 and the comments following Corollary 27.2.2 of Rockafellar (1997).

Fix any $(x^*, y^*) \in \{(x, y), (x', y')\}$. Then $(x^*, y^*) \notin \operatorname{epi} f^{\infty} = (\operatorname{epi} f)^{\infty}$ which implies that there exists $(z^1, z^2) \in \operatorname{epi} f$ and $\lambda > 0$ such that

$$(z^1, z^2) + \lambda(x^*, y^*) \notin \operatorname{epi} f,$$

in other words,

$$f(z^1 + \lambda x^*) > z^2 + \lambda y^*,$$

where $\lambda y^* > 0$. This implies that

$$f(z^1 + \lambda x^*) > z^2 \geq f(z^1)$$

since $(z^1, z^2) \in \operatorname{epi} f$. By taking $(x^*, y^*) = (x, y)$, it yields $f(z^1 + \lambda x) > f(z^1)$, where $z^1 + \lambda x < z^1$. Thus f is not nondecreasing. Similarly, by taking $(x^*, y^*) = (x', y')$, we have $f(z^1 + \lambda x') > f(z^1)$ with $z^1 + \lambda x' > z^1$ and hence f is not nonincreasing. □

The following lemma provides a technical result that is used in the proof of Theorem 5.25.

Lemma A.6. *Suppose that f and g are $\mathbb{R} \cup \{\infty\}$-valued functions on a non-empty set A such that*

$$\left|\inf_{x \in A} f(x)\right| < \infty, \qquad \left|\inf_{y \in A} g(x)\right| < \infty.$$

Then

$$\left|\inf_{x \in A} f(x) - \inf_{x \in A} g(x)\right| \leq \sup_{x \in A} |f(x) - g(x)|. \tag{A.1}$$

Proof. Observe that

$$\inf_{x \in A} f(x) = -\sup_{x \in A}[-f(x)]$$

and

$$\inf_{x \in A} g(x) = -\sup_{x \in A}[-g(x)].$$

We are going prove (A.1) by considering the following two situations for $\inf_{x \in A} f(x) - \inf_{x \in A} g(x)$.

In the situation when $\inf_{x \in A} f(x) - \inf_{x \in A} g(x) \geq 0$, we have

$$\left|\inf_{x \in A} f(x) - \inf_{x \in A} g(x)\right| = \inf_{x \in A} f(x) - \inf_{x \in A} g(x) = \sup_{x \in A}\left[\inf_{y \in A} f(y) - g(x)\right].$$

Then it follows from $\inf_{y \in A} f(y) \leq f(x)$ for any $x \in A$ that

$$\left|\inf_{x \in A} f(x) - \inf_{x \in A} g(x)\right| \leq \sup_{x \in A}[f(x) - g(x)] \leq \sup_{x \in A} |f(x) - g(x)|,$$

and hence (A.1) holds true.

In the situation when $\inf_{x \in A} f(x) - \inf_{x \in A} g(x) < 0$, we have

$$\left|\inf_{x \in A} f(x) - \inf_{x \in A} g(x)\right| = \inf_{x \in A} g(x) - \inf_{x \in A} f(x) = \sup_{x \in A}\left[\inf_{y \in A} g(y) - f(x)\right].$$

Since $\inf_{y \in A} g(y) \leq g(x)$ for any $x \in A$, it follows that

$$\left|\inf_{x \in A} f(x) - \inf_{x \in A} g(x)\right| \leq \sup_{x \in A}[g(x) - f(x)] \leq \sup_{x \in A} |f(x) - g(x)|.$$

This completes the proof of (A.1). □

The lemma below is used in proof of Lemma 4.14, and it is also used in Example 5.10.

Lemma A.7. *Let $x \in \mathbb{R}$. Moreover, let $[b_1, a_1]$ and $[b_2, a_2]$ be two subintervals of $\mathbb{R}$, where $b_1 \leq a_1$ and $b_2 \leq a_2$. Then*

$$C = \{\gamma \in [0,1] \,|\, \exists\, x_1 \in [b_1, a_1], x_2 \in [b_2, a_2] : \gamma x_1 + (1-\gamma)\, x_2 = x\} \tag{A.2}$$

is a convex set in $[0,1]$.

Proof. Suppose that $\gamma^1, \gamma^2 \in C$ and $\mu \in (0,1)$. Then it follows from (A.2) that there exist $x_1^1, x_1^2 \in [b_1, a_1]$ and $x_2^1, x_2^2 \in [b_2, a_2]$ such that

$$\begin{aligned} \gamma^1 x_1^1 + (1-\gamma^1) x_2^1 &= x, \\ \gamma^2 x_1^2 + (1-\gamma^2) x_2^2 &= x. \end{aligned}$$

Let

$$\gamma^* := \mu\gamma^1 + (1-\mu)\gamma^2 \in [0,1].$$

Notice that, if $\gamma^* = 1$, then $\gamma^1 = \gamma^2 = 1$ which means $x_1^1 = x_1^2 = x$. Moreover, if $\gamma^* = 0$, then $\gamma^1 = \gamma^2 = 0$ which means $x_2^1 = x_2^2 = x$. To prove C is convex, it is enough to show that $\gamma^* \in C$. Let

$$x_1^* := \begin{cases} \frac{\mu\gamma^1}{\gamma^*} x_1^1 + \frac{(1-\mu)\gamma^2}{\gamma^*} x_1^2 & \text{if } \gamma^* \in (0,1), \\ b_1 & \text{if } \gamma^* = 0, \\ x & \text{if } \gamma^* = 1, \end{cases}$$

where $\frac{\mu\gamma^1}{\gamma^*}, \frac{(1-\mu)\gamma^2}{\gamma^*} \geq 0$ and

$$\frac{\mu\gamma^1}{\gamma^*} + \frac{(1-\mu)\gamma^2}{\gamma^*} = \frac{\mu\gamma^1 + (1-\mu)\gamma^2}{\gamma^*} = \frac{\gamma^*}{\gamma^*} = 1.$$

In addition, let

$$x_2^* := \begin{cases} \frac{\mu(1-\gamma^1)}{1-\gamma^*} x_2^1 + \frac{(1-\mu)(1-\gamma^2)}{1-\gamma^*} x_2^2 & \text{if } \gamma^* \in (0,1), \\ x & \text{if } \gamma^* = 0, \\ a_2 & \text{if } \gamma^* = 1, \end{cases}$$

where $\frac{\mu(1-\gamma^1)}{1-\gamma^*}, \frac{(1-\mu)(1-\gamma^2)}{1-\gamma^*} \geq 0$ and

$$\frac{\mu(1-\gamma^1)}{1-\gamma^*} + \frac{(1-\mu)(1-\gamma^2)}{1-\gamma^*} = \frac{1-\mu\gamma^1 - (1-\mu)\gamma^2}{1-\gamma^*} = \frac{1-\gamma^*}{1-\gamma^*} = 1.$$

Observe that $x_1^* \in [b_1, a_1]$ and $x_2^* \in [b_2, a_2]$. We are going to show that

$$\gamma^* x_1^* + (1-\gamma^*)x_2^* = x$$

by considering the following three cases of γ^*. If $\gamma^* \in (0,1)$, then

$$\begin{aligned}\gamma^* x_1^* + (1-\gamma^*)x_2^* &= \mu\gamma^1 x_1^1 + (1-\mu)\gamma^2 x_1^2 + \mu(1-\gamma^1)x_2^1 + (1-\mu)(1-\gamma^2)x_2^2 \\ &= \mu(\gamma^1 x_1^1 + (1-\gamma^1)x_2^1) + (1-\mu)(\gamma^2 x_1^2 + (1-\gamma^2)x_2^2) \\ &= \mu x + (1-\mu)x \\ &= x.\end{aligned}$$

If $\gamma^* = 1$, then

$$\gamma^* x_1^* + (1-\gamma^*)x_2^* = 1 \times x + 0 \times a_2 = x.$$

Similarly, if $\gamma^* = 0$, then

$$\gamma^* x_1^* + (1-\gamma^*)x_2^* = 0 \times b_1 + 1 \times x = x.$$

We can conclude that $\gamma^* \in C$, and hence C is convex. □

A.2 Piecewise linear convex function

This section provides a number of basic facts about piecewise linear convex functions. These results are helpful in the study of Sections 5.3 and 5.4.

The following technical result will used in Lemma A.10; it is also used in Lemmas 5.29 and 5.32.

Lemma A.8. *Let f be an $\mathbb{R} \cup \{\infty\}$-valued convex function on a convex set $C \subseteq \mathbb{R}$ such that $f < \infty$ on $[x_1, x_2] \subseteq C$ for some $x_1 < x_2$. Moreover, let h be the $\mathbb{R}$-valued affine function on $\mathbb{R}$ such that $h(x_1) = f(x_1)$ and $h(x_2) = f(x_2)$, in other words,*

$$\begin{aligned}h(x) &= \alpha x + \beta, \\ \alpha &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}, \\ \beta &= f(x_1) - \alpha x_1.\end{aligned}$$

Then $f \le h$ on $[x_1, x_2]$, and $f \ge h$ on $C \backslash (x_1, x_2)$.

Proof. For any $x \in [x_1, x_2]$, there exists $\theta \in [0,1]$ such that $x = \theta x_1 + (1-\theta)x_2$.

Then the convexity of f gives

$$f(x) \leq \theta f(x_1) + (1-\theta)f(x_2) = \theta h(x_1) + (1-\theta)h(x_2) = h(x).$$

Thus $f \leq h$ on $[x_1, x_2]$.

Fix any $x \in C\backslash(x_1, x_2)$. Consider the following two situations. If $x \leq x_1$, then the quantity $\theta \in (0,1]$ such that $x_1 = \theta x + (1-\theta)x_2$ satisfies

$$f(x_1) \leq \theta f(x) + (1-\theta)f(x_2).$$

Then combining this with $h(x_1) = \theta h(x) + (1-\theta)h(x_2)$, it follows that

$$f(x) \geq \frac{f(x_1) - (1-\theta)f(x_2)}{\theta} = \frac{h(x_1) - (1-\theta)h(x_2)}{\theta} = h(x).$$

Similarly, if $x \geq x_2$ then the quantity $\theta \in [0,1)$ such that $x_2 = \theta x_1 + (1-\theta)x$ satisfies

$$f(x_2) \leq \theta f(x_1) + (1-\theta)f(x).$$

This implies

$$f(x) \geq \frac{f(x_2) - \theta f(x_1)}{1-\theta} = \frac{h(x_2) - \theta h(x_1)}{1-\theta} = h(x)$$

because $h(x_2) = \theta h(x_1) + (1-\theta)h(x)$. Thus $f \geq h$ on $C\backslash(x_1, x_2)$. □

The following result shows that a continuous piecewise linear function with nondecreasing slopes is convex.

Lemma A.9. *Let $x_1, \ldots, x_n \in \mathbb{R}$ such that $x_1 \leq \cdots \leq x_n$, and let*

$$h_i(x) = \alpha_i x + \beta_i$$

with $\alpha_i, \beta_i \in \mathbb{R}$ for all $x \in \mathbb{R}$ and $i = 1, \ldots, n-1$. If $\alpha_1 \leq \cdots \leq \alpha_{n-1}$ and h is a continuous piecewise linear function on $[x_1, x_n]$ such that

$$h(x) = h_i(x) \text{ for all } x \in [x_i, x_{i+1}], i = 1, \ldots, n-1,$$

then the following two claims hold true.

1. *For any $i = 1, \ldots, n-1$ and $x \in [x_i, x_{i+1}]$, we have*

$$h(x) = h_i(x) = \max\{h_1(x), \ldots, h_{n-1}(x)\}.$$

2. *The function h is convex.*

Proof. Firstly, we are going to prove the first claim. Fix any $i = 1, \dots, n-1$ and $x \in [x_i, x_{i+1}]$. Clearly, we have $h(x) = h_i(x)$, and it is enough to show

$$h_i(x) = \max\{h_1(x), \dots, h_{n-1}(x)\}. \tag{A.3}$$

For any $1 \le k \le n-2$, we have $h = h_k$ on $[x_k, x_{k+1}]$ and $h = h_{k+1}$ on $[x_{k+1}, x_{k+2}]$, which implies

$$h_k(x_{k+1}) = h(x_{k+1}) = h_{k+1}(x_{k+1}). \tag{A.4}$$

Consider the following two cases. If $x = x_{k+1}$, then $h_k(x) = h_{k+1}(x)$ by (A.4). In the case when $x \neq x_{k+1}$, we have

$$\frac{h_k(x_{k+1}) - h_k(x)}{x_{k+1} - x} = \alpha_k \le \alpha_{k+1} = \frac{h_{k+1}(x_{k+1}) - h_{k+1}(x)}{x_{k+1} - x},$$

and (A.4) implies

$$\frac{h_k(x_{k+1}) - h_k(x)}{x_{k+1} - x} \le \frac{h_k(x_{k+1}) - h_{k+1}(x)}{x_{k+1} - x}.$$

Thus

$$x < x_{k+1} \Longrightarrow h_k(x) \ge h_{k+1}(x)$$

and

$$x > x_{k+1} \Longrightarrow h_k(x) \le h_{k+1}(x).$$

We can conclude that

$$h_i(x) \ge \cdots \ge h_{n-1}(x)$$

and

$$h_1(x) \le \cdots \le h_i(x).$$

Therefore (A.3) holds true. This completes the proof of the first claim.

The first claim implies that

$$h(x) = \max\{h_1(x), \dots, h_{n-1}(x)\} \text{ for all } x \in [x_1, x_n].$$

Then h is convex by Theorem 5.5 of Rockafellar (1997). This completes the proof of the second claim. □

Let $x_1, \dots, x_n \in \mathbb{R}$ such that $x_1 < \cdots < x_n$, and let $f : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ be a continuous convex function such that $f < \infty$ on $[x_1, x_n]$. By connecting $(x_i, f(x_i))$ and $(x_{i+1}, f(x_{i+1}))$ for each $i = 1, \dots, n-1$, we are going to construct a continuous piecewise linear function h on $[x_1, x_n]$ such that $h = f$ on

$\{x_1, \ldots, x_n\}$ as follows. Firstly, define

$$h(x) := f(x) \text{ for all } x = x_1, \ldots, x_n. \tag{A.5}$$

Then for any $i = 1, \ldots, n-1$ and $x \in (x_i, x_{i+1})$, let

$$h(x) := h_i(x) = \alpha_i x + \beta_i \tag{A.6}$$

where h_i is an affine function on $\mathbb{R}$ with

$$\alpha_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i},$$
$$\beta_i = f(x_i) - \alpha_i x_i.$$

This completes the definition of h on $[x_1, x_n]$. Notice that h is real-valued, continuous, piecewise linear, and satisfies

$$h(x) = h_i(x) \text{ for all } x \in [x_i, x_{i+1}], i = 1, \ldots, n-1. \tag{A.7}$$

The following result provides a number of properties of $h_1, \ldots, h_{n-1}$ and h.

Lemma A.10. *The slopes of affine functions $h_1, \ldots, h_{n-1}$ satisfy*

$$\alpha_1 \leq \cdots \leq \alpha_{n-1}. \tag{A.8}$$

Moreover, the function h defined in (A.5)-(A.6) *is real-valued, continuous, piecewise linear, convex, and satisfies $h \geq f$ on $[x_1, x_n]$.*

Proof. We are going to prove (A.8) first. For any $i = 1, \ldots, n-2$, there exists $\theta \in (0,1)$ such that $x_{i+1} = \theta x_i + (1-\theta)x_{i+2}$, and the convexity of f gives

$$f(x_{i+1}) \leq \theta f(x_i) + (1-\theta) f(x_{i+2}).$$

This implies

$$f(x_{i+1}) - f(x_i) \leq (\theta - 1) f(x_i) + (1-\theta) f(x_{i+2}) = (1-\theta)(f(x_{i+2}) - f(x_i)) \tag{A.9}$$

and

$$f(x_{i+2}) - f(x_{i+1}) \geq \theta f(x_{i+2}) - \theta f(x_i) = \theta(f(x_{i+2}) - f(x_i)). \tag{A.10}$$

From (A.9) and $x_{i+1} - x_i = (1-\theta)(x_{i+2} - x_i)$, we have

$$\alpha_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} \leq \frac{(1-\theta)(f(x_{i+2}) - f(x_i))}{(1-\theta)(x_{i+2} - x_i)} = \frac{f(x_{i+2}) - f(x_i)}{x_{i+2} - x_i}.$$

Similarly, combining (A.10) and $x_{i+2} - x_{i+1} = \theta(x_{i+2} - x_i)$, it follows that

$$\alpha_{i+1} = \frac{f(x_{i+2}) - f(x_{i+1})}{x_{i+2} - x_{i+1}} \geq \frac{\theta(f(x_{i+2}) - f(x_i))}{\theta(x_{i+2} - x_i)} = \frac{f(x_{i+2}) - f(x_i)}{x_{i+2} - x_i}.$$

Thus $\alpha_i \leq \alpha_{i+1}$. This completes the proof of (A.8).

Clearly, the function h is real-valued, continuous and piecewise linear. Moreover, combining Lemma A.9 together with (A.8) and (A.7), the function h is convex. For any $i = 1, \ldots, n-1$, we have $h_i(x_i) = f(x_i)$ and $h_i(x_{i+1}) = f(x_{i+1})$, and it follows from Lemma A.8 that $h_i \geq f$ on $[x_i, x_{i+1}]$. Then $h \geq f$ on $[x_1, x_n]$ by (A.7). □

A.3 Set-valued function and random function

This section will start by introducing the notion of *a set-valued function* and the measurability of such type of function. After that, the section will introduce the notion of *a random function* and the measurability of this type of function.

In the remainder of this section, let $d \in \mathbb{N}$. We denote the power set of $\mathbb{R}^d$ by $2^{\mathbb{R}^d}$. For any set X, a function of the form $f : X \to 2^{\mathbb{R}^d}$ is called a *set-valued function*.

We will keep $t = 0, \ldots, T$ fixed in the remainder of this section. The following example considers a set-valued function on $\mathcal{L}_t^d$.

Example A.11. Define

$$f(x) := \left(\mathbb{E}[x^1], \ldots, \mathbb{E}[x^d]\right) + \mathbb{R}_+^d \text{ for all } x = (x^1, \ldots, x^d) \in \mathcal{L}_t^d.$$

Then f is a set-valued function from $\mathcal{L}_t^d$ to $2^{\mathbb{R}^d}$.

In our setting, the concept of a measurable set-valued function from Definition 14.1 of Rockafellar & Wets (2009) can be presented as follows.

Definition A.12. A set-valued function $f : \Omega \to 2^{\mathbb{R}^d}$ is called $\mathcal{F}_t$-measurable if

$$\{\omega \in \Omega \mid f^\omega \cap O \neq \emptyset\} \in \mathcal{F}_t \text{ for all open } O \subseteq \mathbb{R}^d. \tag{A.11}$$

For any set-valued function $f : \Omega \to 2^{\mathbb{R}^d}$, it is easy to see that if the function $\omega \mapsto f^\omega$ is constant on each node in Ω_t, in other words,

$$f^\omega = f^{\omega'} \text{ for all } \omega, \omega' \in \nu \text{ and } \nu \in \Omega_t, \tag{A.12}$$

then this function is $\mathcal{F}_t$-measurable. In the situation when (A.12) holds true, it will sometimes be convenient to denote the common value of f on $\nu \in \Omega_t$

by f^ν. Moreover, for any random variable $z \in \mathcal{L}_t$, we write $z \in f$ if $z(\nu) \in f^\nu$ for all $\nu \in \Omega_t$.

The next example considers a measurable set-valued function on Ω.

Example A.13. Let $x \in \mathcal{L}_t^d$. Define the set-valued function

$$f^\omega := x(\omega) + \mathbb{R}_+^d \text{ for all } \omega \in \Omega.$$

Then f is $\mathcal{F}_t$-measurable because $x \in \mathcal{L}_t^d$.

The example below shows that it is possible that $f : \Omega \to 2^{\mathbb{R}^d}$ is $\mathcal{F}_t$-measurable, but it does not satisfy (A.12).

Example A.14. Let $d = 1, t = 0$, and $\Omega = \{\omega_1, \omega_2\}$. Then $\mathcal{F}_0 = \{\{\omega_1, \omega_2\}, \emptyset\}$. Define $f^{\omega_1} = \mathbb{R}$ and $f^{\omega_2} = \mathbb{R}\backslash\{0\}$. Clearly, we have $f^{\omega_1} \cap O \neq \emptyset$ for all non-empty open $O \subseteq \mathbb{R}$. Moreover, if there exists a non-empty $O \subseteq \mathbb{R}$ such that $f^{\omega_2} \cap O = \emptyset$, then O must be $\{0\}$ which is not open. This means $f^{\omega_2} \cap O \neq \emptyset$ for all non-empty open $O \subseteq \mathbb{R}$. Thus

$$\{\omega \in \Omega \mid f^\omega \cap O \neq \emptyset\} \in \mathcal{F}_0 \text{ for all open } O \subseteq \mathbb{R}.$$

in other words, the function f is $\mathcal{F}_0$-measurable. However, it follows from $f^{\omega_1} \neq f^{\omega_2}$ that (A.12) is not satisfied for $t = 0$ and $d = 1$.

Now, we are going to introduce the notion of a random function and the measurability of such a function. A function $f : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is called a *random function*.

Example A.15. Given $y \in \mathcal{L}_t^d$, the function $f(x) = x \cdot y$ for $x \in \mathcal{L}_t^d$ corresponds to the random function with value

$$f^\omega(x(\omega)) = x(\omega) \cdot y(\omega) \text{ for all } \omega \in \Omega.$$

For any $\omega \in \Omega$, the function f^ω has domain $\mathbb{R}^d$ and range $\mathbb{R} \cup \{\infty\}$. Moreover, since $y \in \mathcal{L}_t^d$, we have for all $\nu \in \Omega_t$ that $f^\omega = f^{\omega'}$ for all $\omega, \omega' \in \nu$.

Observe that if f is a random function, then for any $\omega \in \Omega$ the function f^ω is of the form $f^\omega : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$.

Definition A.16. A random function f is called $\mathcal{F}_t$*-measurable* if the set-valued function $\omega \mapsto \operatorname{epi} f^\omega$ is $\mathcal{F}_t$-measurable.

Notice that, for any random function f, if the function $\omega \mapsto f^\omega$ is constant on each node in Ω_t, in other words,

$$f^\omega = f^{\omega'} \text{ for all } \omega, \omega' \in \nu \text{ and } \nu \in \Omega_t, \tag{A.13}$$

then it is $\mathcal{F}_t$-measurable. In the case when (A.13) holds true, we will sometimes use f^ν to represent the common value of f on $\nu \in \Omega_t$. Observe that the random function in Example A.15 is $\mathcal{F}_t$-measurable.

Bibliography

Abrams, R. A. & Karmarkar, U. S. (1980), 'Optimal multiperiod investment-consumption policies', *Econometrica: Journal of the Econometric Society* **48**(2), 333–353.

Arkin, V. I. & Evstigneev, I. V. (1987), *Stochastic Models of Control and Economic Dynamics*, Academic Press, London.

Atkinson, C. & Quek, G. (2012), 'Dynamic portfolio optimization in discrete-time with transaction costs', *Applied Mathematical Finance* **19**(3), 265–298.

Bensaid, B., Lesne, J.-P., Pagès, H. & Scheinkman, J. (1992), 'Derivative asset pricing with transaction costs', *Mathematical Finance* **2**(2), 63–86.

Benth, F. E., Groth, M. & Lindberg, C. (2010), 'The implied risk aversion from utility indifference option pricing in a stochastic volatility model', *International Journal of Applied Mathematics & Statistics* **16**(M10), 11–37.

Bertsekas, D. P. (2015), *Convex optimization algorithms*, Athena Scientific.

Bichuch, M. (2012), 'Asymptotic analysis for optimal investment in finite time with transaction costs', *SIAM Journal on Financial Mathematics* **3**(1), 433–458.

Bingham, N. H. & Kiesel, R. (2004), *Risk-neutral valuation: Pricing and hedging of financial derivatives*, Springer-Ver1ag.

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *The Journal of Political Economy* **81**(3), 637–654.

Boyle, P. P. & Lin, X. (1997), 'Optimal portfolio selection with transaction costs', *North American Actuarial Journal* **1**(2), 27–39.

Brown, D. B. & Smith, J. E. (2011), 'Dynamic portfolio optimization with transaction costs: Heuristics and dual bounds', *Management Science* **57**(10), 1752–1770.

Cai, Y. (2009), Dynamic programming and its application in economics and finance, PhD thesis, PhD thesis, Stanford University.

Cai, Y., Judd, K. L. & Xu, R. (2013), Numerical solution of dynamic portfolio optimization with transaction costs, Technical report, National Bureau of Economic Research.

Carmona, R. (2009), *Indifference Pricing: Theory and Applications*, Princeton University Press.

Cetin, U. & Rogers, L. (2007), 'Modeling liquidity effects in discrete time', *Mathematical Finance* **17**(1), 15–29.

Clewlow, L. & Hodges, S. (1997), 'Optimal delta-hedging under transactions costs', *Journal of Economic Dynamics and Control* **21**(8), 1353–1376.

Constantinides, G. M. & Zariphopoulou, T. (1999), 'Bounds on prices of contingent claims in an intertemporal economy with proportional transaction costs and general preferences', *Finance and Stochastics* **3**(3), 345–369.

Cutland, N. J. & Roux, A. (2012), *Derivative Pricing in Discrete Time*, Springer Science & Business Media.

Cvitanić, J. & Karatzas, I. (1992), 'Convex duality in constrained portfolio optimization', *The Annals of Applied Probability* **2**(4), 767–818.

Cvitanić, J. & Karatzas, I. (1996), 'Hedging and portfolio optimization under transaction costs: a martingale approach', *Mathematical Finance* **6**(2), 133–165.

Czichowsky, C., Peyre, R., Schachermayer, W. & Yang, J. (2018), 'Shadow prices, fractional Brownian motion, and portfolio optimisation under transaction costs', *Finance and Stochastics* **22**(1), 161–180.

Dai, M. & Yi, F. (2009), 'Finite-horizon optimal investment with transaction costs: a parabolic double obstacle problem', *Journal of Differential Equations* **246**(4), 1445–1469.

Davis, M. H. & Norman, A. R. (1990), 'Portfolio selection with transaction costs', *Mathematics of Operations Research* **15**(4), 676–713.

Davis, M. H., Panas, V. G. & Zariphopoulou, T. (1993), 'European option pricing with transaction costs', *SIAM Journal on Control and Optimization* **31**(2), 470–493.

Deelstra, G., Pham, H. & Touzi, N. (2001), 'Dual formulation of the utility maximization problem under transaction costs', *Annals of Applied Probability* **11**(4), 1353–1383.

Delbaen, F., Kabanov, Y. M. & Valkeila, E. (2002), 'Hedging under transaction costs in currency markets: a discrete-time model', *Mathematical Finance* **12**(1), 45–61.

Dempster, M. A. H., Evstigneev, I. V. & Taksar, M. I. (2006), 'Asset pricing and hedging in financial markets with transaction costs: An approach based on the von Neumann–Gale model', *Annals of Finance* **2**(4), 327–355.

Dumas, B. & Luciano, E. (1991), 'An exact solution to a dynamic portfolio choice problem under transactions costs', *The Journal of Finance* **46**(2), 577–595.

Dynkin, E. B. (1972), 'Stochastic concave dynamic programming', *Mathematics of the USSR-Sbornik* **16**(4), 501–515.

Dynkin, E. B. & Yushkevich, A. A. (1979), *Controlled Markov Processes*, Springer-Verlag, New York.

Edirisinghe, C., Naik, V. & Uppal, R. (1993), 'Optimal replication of options with transactions costs and trading restrictions', *Journal of Financial and Quantitative Analysis* **28**(01), 117–138.

Evstigneev, I. V. & Zhitlukhin, M. V. (2013), 'Controlled random fields, von Neumann–Gale dynamics and multimarket hedging with risk', *Stochastics* **85**(4), 652–666.

Föllmer, H. & Schied, A. (2011), *Stochastic finance: An introduction in discrete time*, 3rd, rev. and extended edn, Walter de Gruyter.

Gale, D. (1956), The closed linear model of production, *in* 'Linear inequalities and related systems', Vol. 38 of *Annals of Mathematics Studies*, Princeton University Press, pp. 285–303.

Gennotte, G. & Jung, A. (1994), 'Investment strategies under transaction costs: the finite horizon case', *Management Science* **40**(3), 385–404.

Guasoni, P. (2002), 'Risk minimization under transaction costs', *Finance and Stochastics* **6**(1), 91–113.

Hobson, D. & Zhu, Y. (2016), 'Optimal consumption and sale strategies for a risk averse agent', *SIAM Journal on Financial Mathematics* **7**(1), 674–719.

Hodges, S. D. & Neuberger, A. (1989), 'Optimal replication of contingent claims under transaction costs', *Review of Futures Markets* **8**, 222–239.

Hugonnier, J., Kramkov, D. & Schachermayer, W. (2005), 'On utility-based pricing of contingent claims in incomplete markets', *Mathematical Finance* **15**(2), 203–212.

Janeček, K. & Shreve, S. E. (2004), 'Asymptotic analysis for optimal investment and consumption with transaction costs', *Finance and Stochastics* **8**(2), 181–206.

Jouini, E. & Kallal, H. (1995), 'Arbitrage in securities markets with short-sales constraints', *Mathematical Finance* **5**(3), 197–232.

Kabanov, Y. M. & Stricker, C. (2001), 'The Harrison–Pliska arbitrage pricing theorem under transaction costs', *Journal of Mathematical Economics* **35**(2), 185–196.

Kallsen, J. & Muhle-Karbe, J. (2010), 'On using shadow prices in portfolio optimization with transaction costs', *The Annals of Applied Probability* **20**(4), 1341–1358.

Kallsen, J. & Muhle-Karbe, J. (2015), 'Option pricing and hedging with small transaction costs', *Mathematical Finance* **25**(4), 702–723.

Kantorovich, L. V. (1960), 'Mathematical methods of organizing and planning production (translated from a report in Russian, dated 1939)', *Management Science* **6**(4), 366–422.

Koopmans, T. C. (1951), Analysis of production as an efficient combination of activities, *in* T. C. Koopmans, ed., 'Activity analysis of production and allocation', Vol. 13 of *Cowles Commission for Researchin Economics Monograph*, Wiley, New York.

Lamberton, D., Pham, H. & Schweizer, M. (1998), 'Local risk-minimization under transaction costs', *Mathematics of Operations Research* **23**(3), 585–612.

Liu, H. (2004), 'Optimal consumption and investment with transaction costs and multiple risky assets', *The Journal of Finance* **59**(1), 289–338.

Löhne, A. & Rudloff, B. (2014), 'An algorithm for calculating the set of superhedging portfolios in markets with transaction costs', *International Journal of Theoretical and Applied Finance* **17**(02), 1450012.

Mania, M. & Schweizer, M. (2005), 'Dynamic exponential utility indifference valuation', *The Annals of Applied Probability* **15**(3), 2113–2143.

Mercurio, F. & Vorst, T. C. F. (1997), Options pricing and hedging in discrete time with transaction costs, *in* M. A. H. Dempster & S. R. Pliska, eds, 'Mathematics of derivative securities', Cambridge University Press, Cambridge, pp. 190–215.

Merton, R. C. (1973), 'Theory of rational option pricing', *The Bell Journal of Economics and Management Science* **4**(1), 141–183.

Monoyios, M. (2003), 'Efficient option pricing with transaction costs', *Journal of Computational Finance* **7**(1), 107–128.

Monoyios, M. (2004), 'Option pricing with transaction costs using a markov chain approximation', *Journal of Economic Dynamics and Control* **28**(5), 889–913.

Musiela, M. & Zariphopoulou, T. (2004), 'A valuation algorithm for indifference prices in incomplete markets', *Finance and Stochastics* **8**(3), 399–414.

Muthuraman, K. (2007), 'A computational scheme for optimal investment–consumption with proportional transaction costs', *Journal of Economic Dynamics and Control* **31**(4), 1132–1159.

Muthuraman, K. & Kumar, S. (2006), 'Multidimensional portfolio optimization with proportional transaction costs', *Mathematical Finance* **16**(2), 301–335.

Øksendal, B. & Sulem, A. (2002), 'Optimal consumption and portfolio with both fixed and proportional transaction costs', *SIAM Journal on Control and Optimization* **40**(6), 1765–1790.

Pennanen, T. (2014), 'Optimal investment and contingent claim valuation in illiquid markets', *Finance and Stochastics* **18**(4), 733–754.

Pennanen, T. & Perkkiö, A.-P. (2012), 'Stochastic programs without duality gaps', *Mathematical Programming* **136**(1), 91–110.

Perrakis, S. & Lefoll, J. (1997), 'Derivative asset pricing with transaction costs: an extension', *Computational Economics* **10**(4), 359–376.

Quek, G. S. H. (2012), Portfolio optimisation and option pricing in discrete time with transaction costs, PhD thesis, Imperial College London.

Rockafellar, R. T. (1974), *Conjugate duality and optimization*, Vol. 16 of *Regional Conference Series in Applied Mathematics*, SIAM.

Rockafellar, R. T. (1997), *Convex analysis*, Princeton Landmarks in Mathematics, Princeton University Press.

Rockafellar, R. T. & Wets, R. J.-B. (2009), *Variational analysis*, Vol. 317 of *Comprehensive Studies in Mathematics*, 3rd edn, Springer.

Roman, S. (2008), *Advanced linear algebra*, Vol. 135 of *Graduate Texts in Mathematics*, 3rd edn, Springer.

Rouge, R. & El Karoui, N. (2000), 'Pricing via utility maximization and entropy', *Mathematical Finance* **10**(2), 259–276.

Roux, A. (2006), European and American options under proportional transaction costs, PhD thesis, University of York.

Roux, A., Tokarz, K. & Zastawniak, T. (2008), 'Options under proportional transaction costs: An algorithmic approach to pricing and hedging', *Acta Applicandae Mathematicae* **103**(2), 201–219.

Roux, A. & Zastawniak, T. (2016), 'American and Bermudan options in currency markets with proportional transaction costs', *Acta Applicandae Mathematicae* **141**(1), 187–225.

Sass, J. (2005), 'Portfolio optimization under transaction costs in the CRR model', *Mathematical Methods of Operations Research* **61**(2), 239–259.

Schachermayer, W. (2002), Optimal investment in incomplete financial markets, *in* H. Geman, D. Madan, S. R. Pliska & T. Vorst, eds, 'Mathematical Finance — Bachelier Congress 2000', Springer Finance, Springer Berlin Heidelberg, pp. 427–462.

Schachermayer, W. (2004), 'The fundamental theorem of asset pricing under proportional transaction costs in finite discrete time', *Mathematical Finance* **14**(1), 19–48.

Shreve, S. E. (2004), *Stochastic calculus for finance II: Continuous-time models*, Vol. 11 of *Springer Finance*, Springer.

Shreve, S. E. & Soner, H. M. (1994), 'Optimal investment and consumption with transaction costs', *The Annals of Applied Probability* **4**(3), 609–692.

Tien, C.-Y. (2011), Mixed stopping times and American options under transaction costs, PhD thesis, The University of York.

von Neumann, J. (1937), Über ein ökonomisches gleichungssystem und eine verallgemeinerung des brouwerschen fixpunktsatzes, *in* 'Ergebnisse eines Mathematischen Kolloquiums', Vol. 8, Franz Deuticke, Leipzig, pp. 73–83. An English translation: (1945) A model of general economic equilibrium, The Review of Economic Studies 13, 1–9.

Zakamouline, V. I. (2005), 'A unified approach to portfolio optimization with linear transaction costs', *Mathematical Methods of Operations Research* **62**(2), 319–343.

Zakamouline, V. I. (2006), 'European option pricing and hedging with both fixed and proportional transaction costs', *Journal of Economic Dynamics and Control* **30**(1), 1–25.